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Coupled thermoelastic shell equations with second sound for high-frequency vibrations of temperature-dependent materials ☆

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Abstract

Guided by the three-dimensional theory of coupled thermoelasticity with second sound, a system of shear-deformable shell equations is consistently derived in invariant, differential and variational forms for the high-frequency vibrations of temperature-dependent materials. The first part of the paper is concerned with a unified variational principle describing the fundamental equations of thermoelasticity. The differential type of variational principle is presented by expressing Hamilton's principle for the thermal part of a thermoelastic region and then combining it with its mechanical part. In the second part, the hierarchic system of non-isothermal shell equations is systematically established by use of the variational principle together with Mindlin's kinematic hypothesis for shells. The system of two-dimensional approximate equations which may take account of all the significant mechanical and thermal effects, including the temperature dependency of material, governs the extensional, thickness-shear, flexural and torsional as well as coupled vibrations of shells of uniform thickness. Lastly, in the third part, emphasis is placed on certain cases involving special material, geometry and kinematics. Besides, a theorem is devised so as to enumerate the initial and boundary conditions sufficient for the uniqueness in solutions of the system of non-isothermal shell equations.

Keywords: Thermoelastic shell vibrations; Variational principle; Uniqueness

1. Introduction

Usually, temperature-dependent and/or time-dependent and even moisture-dependent materials are of widespread use in structural elements of diverse branches of today's advanced technology. Hence, for this type of elements which are essentially subjected to thermal or combined thermo-mechanical loadings over a wide range of frequencies, the lower-order thermoelastic equations are a subject of considerably increasing importance. To establish the thermoelastic equations, the conventional uncoupled theories of thermoelasticity are justified in most cases by their very good corroboration of several experimental investigations,

☆ Dedicated to Professor Bruno A. Boley on his seventy-fifth birthday

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Nomenclature

θ^i	geodesic normal coordinates
t^{ij}	components of the stress tensor
f^i	components of the body force vector
ρ	mass density
u^i, $a^i(=\ddot{u}^i)$	components of the displacement and acceleration vectors
h^i	components of the heat flux vector
$\not{f}$	heat source per unit mass
Θ_0, Θ	reference temperature, temperature increment
η	entropy density
e_{ij}	components of the linear strain tensor
e_i	components of the thermal field vector
G, F	thermoelastic potential, dissipation function
τ_0	relaxation parameter
k^i	components of the conductivity tensor
λ^{ij}, c^{ijkl}	material constants
α	thermal expansion coefficient
$2\hbar$	thickness of thermoelastic shell

however they become inadequate in certain cases indicated, for instance, by Boley and Weiner (1960). Thus, the fusion of both the fields of elasticity and heat conduction, that is, the coupled theory of dynamic thermoelasticity is compulsory in investigating the thermomechanical response of heated elements in which the thermoelastic dissipation is of primary interest. This classical coupled theory which predicts the paradox of instantaneous propagation for thermal signals seems to be unrealistic from the physical point of view. Accordingly, to eliminate the paradox of infinite heat speed, the coupled theory is modified by introducing a thermal relaxation parameter into Fourier's equation of heat conduction and the so-called theories of thermoelasticity with second sound or hyperbolic thermoelasticity were proposed. The modified equation of heat conduction or Cattaneo's (1958) equation which can be traced to Maxwell (1867) and also Jeffrey's type equation of heat conduction suggested by Joseph and Preziosi (1989, 1990) assure, unlike their classical counterparts, a finite speed for heat propagation. Noteworthy is that the hyperbolic thermoelasticity theories are in need of further modification due to their some unphysical results, as was shown by Solomon et al. (1985). [1] The effect of thermal relaxation is significant in certain cases (Chandrasekharaiah, 1986, 1998; Li, 1992; Hetnarski and Ignaczak, 2000), though it is very small and negligible in many engineering applications. By applying the coupled theories of thermoelasticity, a large number of investigations was performed so as to predict the physical behavior of heated structural elements and were reported by the state-of-the-art accounts (Hetnarski, 1986–1989; Lukasiewicz, 1989; Noda, 1986, 1991; Tauchert, 1987, 1991; Noor and Burton, 1992; Thornton, 1993, 1996, 1997; Noor et al., 1996). The accounts were concentrated almost entirely, omitting the effect of thermal relaxation, on specific cases dealing with either the static behavior or the dynamic behavior of heated structural elements at low-frequencies, and in particular, the unified thermoelastic equations, including the effect of thermal relaxation, seem to be unavailable for the high-frequency vibrations of heat-sensitive shells or plates.

Looking back on the fundamental equations of a non-polar, non-local and non-relativistic elastic medium subjected to thermo-mechanical loading, they may be grouped as the divergence, gradient and

[1] We are grateful to the reviewer who brought this reference to our attention.

constitutive equations for the mechanical and thermal fields with the appropriate boundary and initial conditions which are supplementary to them. The divergence equations which are originally established in an integral form and the rest of the fundamental equations are almost always expressed in differential form. However, some or all the fundamental equations may be alternatively stated in variational form through the Euler–Lagrange equations of certain integral or differential types of variational principles with their well-known features. A number of variational principles in thermoelasticity (Biot, 1956, 1984; Herrmann, 1963; Nickell and Sackman, 1968; Dhaliwal and Sherief, 1980; Batra, 1989; Li, 1992) are contrived intutively or by a trial-and-error method, and also, either by a purely mathematical method (Gurtin, 1972) or through a general principle of physics (Altay and Dökmeci, 1996; Argyris and Tenek, 1997). Carlson (1972) and Keramides (1983) reported a comprehensive review of variational principles in thermoelasticity, as did recently Altay and Dökmeci (1996). By introducing a thermal field vector for the gradient of temperature increment, a new concept analogous to the electric field vector for the gradient of electric potential in piezoelectricity, Altay and Dökmeci (1996) derived certain variational principles by extending the principle of virtual work for the coupled discontinuous thermal and mechanical fields. Likewise, a Hamiltonian type-variational principle is now presented for the coupled theory of thermoelasticity with second sound.

At low frequencies in which the wavelength is large as compared with the thickness, the vibrations of shells/plates are most commonly encountered in conventional engineering applications, whereas the high frequency vibrations where the wavelength is of the order of magnitude or smaller than the thickness are finding applications in advanced technology (Thornton, 1996, 1997). Investigations are abundant for the low frequency vibrations of shells/plates with various shapes and materials (Leissa, 1973; Noor, 1990) and are rather scanty for the high-frequency vibrations of shells/plates (Junger and Feit, 1972; Tovstik, 1992; Ivanova, 1998; Altay and Dökmeci, 1998). A unified treatment of high frequency vibrations and waves in solids was reported by Truell and Elbaum (1962), including non-linear and anharmonic effects, thermoelastic and electrical effects and some experimental results. As for the vibrations of structures, Ekstein (1945) seems to be the first to analyze the high frequency vibrations of a thin crystal plate by a procedure based on the series expansion methods of Cauchy (1829) and Poisson (1829). By the same procedure together with the integral method of Kirchhoff (1850), Mindlin (1955) studied various types of high-frequency vibrations of elastic plates. Using Mindlin's celebrated method of reduction (Mindlin, 1968, 1989), a number of authors (Tasi and Herrmann, 1964; Tiersten, 1969; Mindlin, 1972, 1989; Nikodem and Lee, 1974; Yong et al., 1993; Dökmeci, 1972, 1974; Altay and Dökmeci, 1996, 1997, and references cited therein) investigated the high-frequency vibrations of beams, plates and shells. On the other hand, based on the variational asymptotic method (Berdichevsky, 1979; Berdichevsky and Lee, 1980) examined the high-frequency response of shells, as did most recently Chau (1997) and Altay and Dökmeci (1998). Besides, for the vibrations of shells, all the available results, computational and experimental, were collected by Leissa (1973). Along this line, many significant equations and results were provided by Soedel (1994) and Pilkey (1994) and a comprehensive review, including background information, was recently reported by Steele et al. (1995).

Yawing now to a severe thermal environment in which the properties of some materials may largely change with increasing temperature (Touloukian, 1970, 1973a,b, 1975), the thermomechanical response of structural elements was reported to be significantly effected by the temperature-dependent properties of materials (Noda (1986, 1991); Thornton (1992, 1993, 1996, 1997); and Noor et al. (1996) for a review of the subject). The analysis of thermomechanical response is becoming increasingly important whenever large ranges of temperature are involved. Nevertheless, only a few specific cases which are numerically treated are available at low-frequencies, due to an increasing number of elasticities for temperature-dependent properties of materials. Realizing the importance of elevated temperature but not elucidating fully the role of thermal effect, Bolotin (1963) derived the basic equations of shells and plates subjected to a linear temperature distribution. The thermal stresses were analyzed for an orthotropic cylinder (Tauchert, 1976a,b; Kalam, 1981), a thick cylindrical shell and tube (Jekot, 1986), a hollow sphere (Tauchert, 1976b; Kamiya, 1980), a hollow cylinder (Kamiya and Kameyama, 1981) and a conical shell (Jianping and Harik, 1991), having temperature-dependent material. A description of thermally induced vibrations in structures, including a review of the past research was given (Thornton and Foster, 1992; Thornton, 1997). In view of the aforementioned reviews, the equations with some applications for thermally induced vibrations in plates were reported (Tomar and Gupta, 1984; Sumi and Sugano, 1997; Altay and Dökmeci, 1997), whereas those in shells seems to be unavailable.

Inspired by the work of Mindlin (1972), the aim of this paper is threefold: (i) to deduce a unified variational principle from Hamilton's principle by expressing it for the thermal part of a thermoelastic region and then combining it with its mechanical part so as to describe the fundamental equations of a coupled theory of thermoelasticiy with second sound; by use of the variational principle (ii) to establish a hierarchic system of two-dimensional, shear-deformable thermoelastic shell equations in invariant, variational and differential forms for the high-frequency vibrations of temperature-dependent materials; and then (iii) to deal with some special cases in the hierarchic system of thermoelastic shell equations and also with the initial and boundary conditions sufficient for the uniqueness in solutions of a linearized version of the equations of thermoelastic shell of uniform thickness.

Looking back on the appropriate notation to be used in the development of invariant shell equations in the remaining of this section, the fundamental equations of a coupled theory of thermoelasticity with second sound are summarized in differential form in the next section. To express the fundamental equations of thermoelasticity in variational form, Hamilton's principle is stated for a regular thermoelastic region with no singularities of any type and then, with the aid of the thermal field vector, a two-field variational principle is derived which leads to the thermal divergence equations and the associated natural boundary conditions for the thermoelastic region. This variational principle is combined with a variational principle for the mechanical part and hence a differential type of unified variational principle is established which yields the fundamental equations of thermoelasticity as its Euler–Lagrange equations. Section 3 is concerned with certain preliminaries and results from the differential geometry of a surface, for ease of quick reference, and with the geometry of a thermoelastic shell of uniform thickness. Besides, the kinematics of thermoelastic shell is described for the high-frequency vibrations and also, the distributions of strain and thermal field are obtained. In the next three sections, emphasis is placed on the derivation of a hierarchic system of two-dimensional, shear-deformable shell equations in invariant differential and invariant variational forms. Section 4 contains the resultants of mechanical and thermal field quantities and the associated constitutive relations for the thermoelastic shell having temperature-dependent materials. By use of the unified variational principle together with the kinematic hypothesis, the hierarchic system of shell equations is systematically deduced from the three-dimensional fundamental equations of thermoelasticity in Section 5. Some of the special cases involving material properties, kinematics and geometry are taken up in Section 6. Besides, a theorem of uniqueness is devised so as to enumerate the boundary and initial conditions sufficient for the uniqueness in solutions of a hierarchic system of fully linearized equations of thermoelastic shell. Some concluding remarks are drawn in the last section.

1.1. Notation

Yearning for its versatility to the differential geometry of a surface, familiar tensor notation is freely used in a three-dimensional Euclidean space Ξ (Ericksen, 1960). Accordingly, Einstein's summation convention is implied for all repeated Latin indices with the range 1, 2, 3 and Greek indices with the range 1, 2, unless the indices (subscripts or superscripts) are enclosed within parentheses. In a fixed, right-handed system of geodesic normal coordinates θ^i (Synge and Schild, 1949) of the space Ξ, Latin indices are assigned to space tensors and Greek indices to surface tensors. A superposed dot stands for time differentiation and a comma for partial differentiation with respect to the indicated space coordinate, and also, a semicolon and a colon for covariant differentiation with respect to the space coordinates, using the space and surface metrics, respectively. Further, an asterisk is used to denote the prescribed quantities and an overbar to indicate the field quantities referred to the base vectors of a reference surface A. In the space Ξ, a finite and bounded, regular region (Kellogg, 1946) is indicated by $\Omega(t)$ with its boundary surface $\partial\Omega$ and closure $\overline{\Omega}(=\Omega\cup\partial\Omega)$ at time t. The Cartesian product of the region Ω and the time interval $T=[t_0,t_1)$, where $t_1>t_0$ may be infinity, is denoted by ΩXT, the thickness interval by $Z=[-\hbar,\hbar]$ in which $2\hbar$ is the thickness of shell and the functions with derivatives of order up to and including (m) and (n) with respect to the space coordinates and time by C_{mn}.

2. Fundamental equations of thermoelasticity in differential and variational forms

In the θ^i system of geodesic normal coordinates, consider a finite and bounded, regular thermoelastic region of space $\Omega+\partial\Omega$ with its boundary surface $\partial\Omega$ and closure $\overline{\Omega}(=\Omega\cup\partial\Omega)$ at time $t=t_0$. The complementary regular subsurfaces of thermoelastic region are indicated by $(\partial\Omega_u,\partial\Omega_t)$ and $(\partial\Omega_\theta,\partial\Omega_\hbar)$, that is, $\partial\Omega_u\cup\partial\Omega_t=\partial\Omega_\theta\cup\partial\Omega_\hbar=\partial\Omega$ and $\partial\Omega_u\cap\partial\Omega_t=\partial\Omega_\theta\cap\partial\Omega_\hbar=\phi$. The unit outward vector normal to the boundary surface is denoted by n_i. The domain of definitions for the mechanical and thermal fields is represented by $\overline{\Omega}XT$ where T is the time interval $[t_0,t_1)$. The motions of thermoelastic region are governed by the well-established fundamental equations of thermoelasticity in differential form (Boley and Weiner, 1960). The fundamental equations may be grouped as the divergence, gradient and constitutive equations and the boundary and initial conditions to supplement them. For completeness and ease of reference, the fundamental equations of thermoelasticity with second sound (Tamma and Namburu, 1997) are summarized below for a non-polar, non-local and non-relativistic elastic medium with temperature-dependent material properties.

Divergence equations (the stress equations of motion and the equation of heat conduction)

$$L_m^j = t^{ij}_{;i} + \rho f^j - \rho a^j = 0 \quad \text{in } \overline{\Omega}XT, \tag{1a}$$

$$\varepsilon_{ijk} t^{jk} = 0 \quad \text{in } \overline{\Omega}XT, \tag{1b}$$

$$L_t = h^i_{;i} + \rho \not{f} + \Theta_0 \dot{\eta} = 0 \qquad \text{in } \overline{\Omega}XT \tag{2}$$

with the definitions

t^{ij}	symmetric components of the stress tensor
ρ	mass density
f^i	components of the body force vector
u^i, a^i	components of the displacement vector and acceleration vector $(=\ddot{u}^i)$
ε_{ijk}	components of the alternating tensor
and	
h^i	components of the heat flux vector
$\not{f}$	heat source unit mass
Θ_0	constant, positive, reference temperature; temperature of natural state of zero stress and strain
η	entropy density

Gradient equations (the strain–displacement relations and the thermal field-temperature increment relations)

$$L_{ij}^m = e_{ij} - \tfrac{1}{2}\left(u_{i;j} + u_{j;i}\right) = 0 \quad \text{in } \overline{\Omega}XT, \tag{3}$$

$$L_i^t = -(e_i + \Theta_{,i}) = 0 \quad \text{in } \overline{\Omega}XT, \tag{4}$$

with the definitions

e_{ij} components of the linear strain tensor
e_i components of the thermal field vector
Θ temperature increment from the reference temperature $\Theta_0, \Theta \ll \Theta_0$

Constitutive relations (for the components of stress tensor, heat flux and the entropy density)

$$L_{\text{mc}}^{ij} = t^{ij} - \frac{1}{2}\left(\frac{\partial G}{\partial e_{ij}} + \frac{\partial G}{\partial e_{ji}}\right) = 0 \quad \text{in } \overline{\Omega}XT \tag{5}$$

and

$$L_{\text{tc}}^{i} = h^i + \frac{\partial G}{\partial e_i} = 0 \quad \text{in } \overline{\Omega}XT, \tag{6}$$

$$L_{\text{tc}} = \eta + \frac{\partial G}{\partial \Theta} = 0 \quad \text{in } \overline{\Omega}XT. \tag{7}$$

Here, the thermoelastic potential G is expressed in terms of the free energy function B and the dissipation function F (Mindlin, 1974), namely

$$G(e_{ij}, e_i, \Theta) = B(e_{ij}, \Theta) - F(e_i). \tag{8}$$

A quadratic form of the thermoelastic potential is defined by

$$B = \tfrac{1}{2}\left(c^{ijkl} e_{ij} e_{kl} - \rho C_v \Theta_0^{-1} \Theta^2\right) - \lambda^{ij} e_{ij} \Theta, \tag{9}$$

$$F = \tfrac{1}{2} k^{ij} e_i e_j - \beta^i e_i; \quad \beta^i = \tau_0 \dot{h}^i. \tag{10}$$

By use of Eqs. (7)–(9), the linear versions of constitutive relations are given in the form

$$L_{\text{mcl}}^{ij} = t^{ij} - \left(c^{ijkl} e_{kl} - \lambda^{ij} \Theta\right) = 0 \tag{11}$$

and

$$L_{\text{tcl}}^{i} = h^i - (k^{ij} e_j - \tau_0 \dot{h}^i) = 0, \tag{12}$$

$$L_{\text{tcl}} = \eta - (\alpha \Theta + \lambda^{ij} e_{ij}) = 0. \tag{13}$$

In the above equations, c^{ijkl} stands for the second order elastic constants measured at constant field and temperature, λ^{ij} for the thermal stress constants relating an increase in temperature to a stress at constant strain or field, k^{ij} for the positive-semidefinite conductivity tensor, $\alpha = \rho C_v \Theta_0^{-1}$ for the linear thermal expansion coefficient, ρC_v for the specific heat per unit volume and τ_0 for a relaxation parameter (nonnegative constant). The relaxation parameter physically signifies the initiation of heat flow after a temperature gradient is imposed. Thus, the anomaly of infinite speed of heat propagation is abrogated. Moreover, the usual symmetry relations of the form

$$c^{ijkl} = c^{jikl} = c^{klij}, \quad \lambda^{ij} = \lambda^{ji}, \quad k^{ij} = k^{ji} \tag{14}$$

is recorded. In the θ^i-system of geodesic normal coordinates where $g^{\alpha 3} = 0$ and $g^{33} = 1$, the constitutive relations (4) take the form,

$$L_{\rm mc}^{\alpha\beta} = t^{\alpha\beta} - \left(c^{\alpha\beta\delta\nu}e_{\delta\nu} + c^{\alpha\beta33}e_{33} - \lambda^{\alpha\beta}\Theta\right) = 0,$$
$$L_{\rm mc}^{\alpha3} = t^{\alpha3} - \left(c^{\alpha3\beta3}e_{\beta3} - \lambda^{\alpha3}\Theta\right) = 0, \tag{15}$$
$$L_{\rm mc}^{33} = t^{33} - \left(c^{33\alpha\beta}e_{\alpha\beta} + c^{3333}e_{33} - \lambda^{33}\Theta\right) = 0$$

for an anisotropic medium having elastic symmetry with respect to the surface $\theta^3 =$ constant (Green and Zerna, 1954), that is,

$$c^{\alpha\beta\nu3} = c^{\nu3\alpha\beta} = c^{\alpha333} = c^{33\alpha3} = 0. \tag{16}$$

For an isotropic medium, the material constants (14) take the form,

$$c^{ijkl} = \lambda g^{ij}g^{kl} + \mu\left(g^{ik}g^{jl} + g^{il}g^{jk}\right), \quad \lambda^{ij} = \alpha g^{ij}, \quad k^{ij} = kg^{ij} \tag{17}$$

in which λ and μ indicate Lame's constants, k is the thermal conductivity and g^{ij} are the components of metric tensor. In accordance with Eq. (17), the constitutive relations become much simpler and the number of material constants are finally reduced to $5(\lambda, \mu, \alpha, \tau_0, k)$ from Eq. (35).

Whenever a thermal field exists, the elasticities of material are essentially dependent on temperature (Miskioğlu et al., 1981; Noda, 1991) and their dependency which largely changes with increasing temperature is reported (e.g., Japan Society of Mechanical Engineers, 1980). Of the elasticities of material, the density, Lamé's constants and the thermal conductivity usually decrease, while the coefficient of thermal expansion increases, with rising temperature. The temperature dependence of material properties is expressed as functions of a trinomial form of temperature increment (Pich, 1981), namely

$$\left(c^{ijkl}, \lambda^{ij}, k^{ij}, \alpha\right) = \sum_{n=0}^{N=2}\left(c_n^{ijkl}, \lambda_n^{ij}, k_n^{ij}, \alpha_n\right)\Theta^n \quad \text{in } \overline{\Omega}XT \tag{18}$$

and the rest of material properties are taken to be constant. Due to the variation of thermal conductivity (18), the equation of heat conduction (2) becomes non-linear as discussed by Carslaw and Jaeger (1959). Nevertheless, this is almost always overlooked in the literature and the equation of heat conduction is assumed, to a reasonable approximation, to be linear. By use of Eq. (18) with the linear temperature dependence of isotropic materials, $N = 1$, only a few studies were reported on the thermally induced vibrations of elastic plates (Tomar and Gupta, 1984; Adeniji-Fashola and Oyediran, 1988; Altay and Dökmeci, 1997). However, the thermally induced vibrations of shells having temperature-dependent material are in need of further investigations.

Fundamental differential equations. The aforementioned equations (1a,b)–(7) comprise the 23 equations of the thermoelasticity with second sound governing the 23 dependent variables $\left(u_i, t^{ij}, e_{ij}; h^i, e_i, \eta, \Theta\right)$ which are the functions of space coordinates and time. The uniqueness in solutions of the initial-mixed boundary value problems defined by the fundamental equations (1a,b)–(7) was examined (Dhaliwal and Sherief, 1980). Thus, a set of initial and boundary conditions sufficient for the uniqueness is stated as follows:

Boundary and initial conditions

$$L_{*m}^{j} = t_*^j - n_i t^{ij} = 0 \quad \text{on } \partial\Omega_t XT, \tag{19}$$

$$L_i^{*m} = u_i^* - u_i = 0 \quad \text{in } \partial\Omega_u XT \tag{20}$$

and

$$L_h^* = h_* - n_i h^i = 0 \quad \text{on } \partial\Omega_h XT, \tag{21}$$

$$L_t^* = -(\Theta - \Theta_*) = 0 \quad \text{on } \partial\Omega_\theta XT, \tag{22}$$

where

t^i components of the traction vector $(= n_j t^{ji})$

h normal components of the heat flux vector across the boundary surface $(= n_i h^i)$ and the initial conditions of the form,

$$L_i^{*om} = u_i(\theta^j, t_0) - \alpha_i^*(\theta^j) = 0 \quad \text{in } \Omega(t_0) \tag{23}$$

$$L_i^{*ob} = \dot{u}_i(\theta^j, t_0) - \beta_i^*(\theta^j) = 0 \quad \text{in } \Omega(t_0) \tag{24}$$

and

$$L_{\text{ot}}^* = \Theta(\theta^i, t_0) - \gamma_*(\theta^i) = 0 \quad \text{in } \Omega(t_0) \tag{25}$$

where α^*, β^* and γ_* are prescribed functions.

2.1. Hamilton's principle

Now, the aforementioned fundamental equations of thermoelasticity with second sound in differential form are expressed in variational form, as the Euler–Lagrange equations of a unified variational principle. To establish the unified variational principle, Hamilton's principle which is a powerful and elegant tool used successfully in continuum physics is used as a starting point for the thermoelastic medium. Hamilton (1834, 1835) principle is originally deduced from D'Alembert's principle by means of an integration over time for a discrete mechanical system and later it is extended by Kirchhoff (1876) for a continuous medium. The application of Hamilton's principle to a finite domain of medium always leads to a variational principle, either an integral type for the case when the non-conservative forces are absent or a differential type (Tabarrok and Rimrott, 1994), that generates only the divergence equations and the associated boundary conditions of medium. In such a variational principle, the variations of each of the field variables are independent (unconstrained) within the domain and are constrained to vanish at the time t_0 and t_1 throughout the domain and its boundary. Hamilton's principle (Goldstein, 1965) is extensively used in solid and fluid mechanics, thermodynamics and electroelasticity (see Kotowski (1992) and Dökmeci (1988) and references cited therein). By applying this principle and modifying it through the dislocation potentials and Lagrange undetermined multipliers, a unified variational principle is recorded whose the Euler–Lagrange equations are shown to be the fundamental equations of thermoelasticity.

A generalized version of Hamilton's principle is proposed for the thermoelastic region, namely

$$\delta L_{\text{H}}\{\wedge\} = \delta L_{\text{m}}\{\wedge_{\text{m}}\} + \delta L_{\text{t}}\{\wedge_{\text{t}}\} = 0 \tag{26}$$

with the denotations of the form,

$$\delta L_{\text{m}}\{\wedge_{\text{m}}\} = \delta \int_T L_{\text{m}\alpha}^{\alpha}\, \text{d}t + \int_T \delta^* W_{\text{m}}\, \text{d}t, \quad \wedge_{\text{m}} = \{u_i\}, \tag{27}$$

$$\delta L_{\text{t}}\{\wedge_{\text{t}}\} = \delta \int_T L_{\text{t}\alpha}^{\alpha}\, \text{d}t + \int_T \delta^* W_{\text{t}}\, \text{d}t, \quad \wedge_{\text{t}} = \{\Theta\} \tag{28}$$

where

$$L_{\text{m}1}^1 = \int_\Omega (K - B)\text{d}V, \qquad L_{\text{m}2}^2 = \int_\Omega \rho f^i u_i\, \text{d}V, \qquad \delta^* W_{\text{m}} = \int_{\partial\Omega} t_*^i \delta u_i\, \text{d}S, \tag{29}$$

$$L^1_{t1} = \int_\Omega (F - B)\mathrm{d}V, \qquad L^2_{t2} = \int_\Omega \Big[(\rho\not{\!f} - \eta)\Theta - \eta\Theta_0\dot{\Theta} \Big]\mathrm{d}V, \qquad \delta^* W_t = \int_{\partial\Omega} h_* \delta\Theta \, \mathrm{d}S \tag{30}$$

and the kinetic energy density K as

$$K = \tfrac{1}{2}\rho\dot{u}^i\dot{u}_i. \tag{31}$$

Here, subscripts (m) and (t) indicate the quantities which belong to the mechanical and thermal parts of Hamilton's principle. The mechanical part of Hamilton's principle is well-known (e.g., Dökmeci, 1988) and hence the thermal part of Hamilton's principle is now considered. Taking pertinent variations in the thermal part, Eq. (26) and then leaving out the variation of mass, namely $\delta(\rho\,\mathrm{d}V) = 0$ due to the axiom of conservation of mass, making use of the interchangeable nature of variation and differentiation or integration and integrating by parts with respect to time, one readily arrives at the variational equation of the form,

$$\delta L_t\{\wedge_t\} = \int_T \mathrm{d}t \int_\Omega \left\{ \left[-\left(\frac{\partial B}{\partial \Theta} + \eta \right) + (\rho\not{\!f} + \dot{\eta}\Theta_0) \right] \delta\Theta + \frac{\partial F}{\partial e_i}\delta e_i \right\} \mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega} h_* \delta\Theta \mathrm{d}S - \int_\Omega \eta\Theta_0\delta\Theta \Big|_T \mathrm{d}V. \tag{32}$$

Recalling the constitutive relations (5)–(9) and imposing the condition that all the variations of temperature increment vanish at $t = t_0$ and $t = t_1$, namely

$$\delta\Theta = 0 \quad \text{at } \Omega(t_0) \text{ and } \Omega(t_1) \tag{33}$$

in Eq. (31) and then substituting Eq. (4), one reads

$$\delta L_t\{\Theta\} = \int_T \mathrm{d}t \int_\Omega \Big[(\rho\not{\!f} + \dot{\eta}\Theta_0)\delta\Theta - h^i \delta\Theta_{,i} \Big] \mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega} h_* \delta\Theta \, \mathrm{d}S. \tag{34}$$

By use of the divergence theorem for the regular thermoelastic region $\Omega + \partial\Omega$, this equation takes the form

$$\delta L_t\{\Theta\} = \int_T \mathrm{d}t \int_\Omega (h^i_{;i} + \rho\not{\!f} + \dot{\eta}\Theta_0)\delta\Theta \, \mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega} (h_* - n_i h^i)\delta\Theta \, \mathrm{d}S. \tag{35}$$

Due to the arbitrary and independent volumetric and surface variations of the admissible state $\wedge_t = \{\Theta\}$, this equation generates the equation of heat conduction and the natural boundary conditions of heat fluxes, as its Euler–Lagrange equations.

2.2. Unified variational principle

The one-field variational principle which recovers the one deduced from the principle of virtual work (Altay and Dökmeci, 1996) is subjected to condition (33) and the remaining equations of thermoelastic medium, that is, Eqs. (4), (6), (7), (22) and (25), as its constraints. This Hamiltonian type variational principle can be readily used to obtain approximate direct solutions to the initial-boundary value problems of thermoelasticity provided that the admissible state satisfies the constraint conditions. On the other hand, in many cases, it is desirable for the approximate (trial or coordinate) functions to satisfy as few constraint conditions as possible, in other cases, it is imperative that the approximating functions do not satisfy some of the constraint conditions. In fact, veering a variational principle with constraints into ones without that is a classical one and a variety of methods is available in removing the constraint conditions (Finlayson and Scriven, 1967). Of the methods to remove constraint conditions, the dislocation potentials and Lagrange undetermined multipliers are used in deriving a variational principle without constraints in thermoelasticity

(Altay and Dökmeci, 1996). In a similar manner, a slightly generalized version of this variational principle may be readily formulated so as to incorporate the effect of second sound by extending the variational principle (35) together with the mechanical part of Hamilton's principle (27). The resulting differential variational principle is recorded, to render the paper self-contained, as an assertion of the form,

$$\delta L\{\wedge_{\mathbf{M}} \cup \wedge_{\mathbf{T}}\} = \delta L_{\mathbf{M}}\{\wedge_{\mathbf{M}}\} + \delta L_{\mathbf{T}}\{\wedge_{\mathbf{T}}\} = \mathbf{0}, \tag{36}$$

where

$$\delta L_{\mathrm{M}}\{\wedge_{\mathrm{M}}\} = \int_T \mathrm{d}t \int_\Omega \left(L^i_{\mathrm{m}} \delta u_i + L^{\mathrm{m}}_{ij} \delta t^{ij} + L^{ij}_{\mathrm{mc}} \delta e_{ij} \right) \mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega_t} L^i_{*\mathrm{m}} \delta u_i \mathrm{d}S + \int_T \mathrm{d}t \int_{\partial\Omega_u} L^{*\mathrm{m}}_i n_j \delta t^{ij} \,\mathrm{d}S, \tag{37}$$

$$\delta L_{\mathrm{T}}\{\wedge_{\mathrm{T}}\} = \int_T \mathrm{d}t \int_\Omega \left(L_{\mathrm{t}} \delta\Theta + L^t_i \delta h^i + L^i_{tc} \delta e_i + L_{tc} \delta\eta \right) \mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega} L^*_h \delta\Theta \,\mathrm{d}S \tag{38}$$

and the admissible states of the form,

$$\wedge_{\mathrm{M}} = \left\{ u_i \in C_{12}, e_{ij} \in C_{00}, t^{ij} \in C_{10} \right\}, \tag{39}$$

$$\wedge_{\mathrm{T}} = \{ \Theta \in C_{11}, e_i \in C_{10}, h^i \in C_{11}, \eta \in C_{10} \} \tag{40}$$

in terms of the denotations (1)–(7) and (19)–(22).

Unified variational principle

Let $\Omega + \partial\Omega$ be a regular, finite and bounded thermoelastic region with its boundary surface $\partial\Omega (= \partial\Omega_u \cup \partial\Omega_t, \partial\Omega_u \cap \partial\Omega_t = \phi)$ and closure $\overline{\Omega}$ in the Euclidean space Ξ. Then, of all the admissible states $\wedge (= \wedge_{\mathrm{M}} \cup \wedge_{\mathrm{T}})$ which satisfy the initial conditions (23)–(25) and the symmetry of stress tensor (1b) as well as the usual existence, continuity and differentiability conditions of field variables, *if and only if*, that admissible state which satisfies the divergence equations (1a) and (2), the gradient equations (3) and (4), the constitutive relations (5)–(7) and the natural boundary conditions (19)–(21) is determined by the seven field variational principle of the form,

$$\delta L\{\wedge\} = 0, \wedge = \wedge_{\mathrm{M}} \cup \wedge_{\mathrm{T}} \tag{41}$$

as it Euler–Lagrange equations. Conversely, if the equations are identically satisfied, the seven field variational principle is met, and thus it is verified.

3. Geometry and kinematics of shell

3.1. Geometry

With reference to the θ^i system of geodesic normal coordinates of the Euclidean space Ξ, the region of thermoelastic shell $V + S$ called the shell space with its boundary surface S and closure $\overline{V} (= V \cup S)$ is taken to be bounded by the lateral surface S_{e}, the lower face S_{lf} and the upper face S_{uf}. The lateral or edge surface $S_{\mathrm{e}} (= S \cap S_{\mathrm{f}}, S_{\mathrm{f}} = S_{\mathrm{uf}} \cup S_{\mathrm{lf}})$ is a right cylindrical surface with generators perpendicular to the midsurface (reference surface) A of thermoelastic shell, and it intersects the reference surface along a Jordan curve C. An outward unit vector normal to the edge surface is denoted by v_i and that to the faces is designated by n_i. The θ^α curves are situated on the reference surface and the θ^3 axis is chosen positively upward, that is, $\theta^3 = 0$ stands for the reference surface and $\theta^3 = \hbar$ and $\theta^3 = -\hbar$ for the upper and lower faces, respectively (Fig. 1). Mathematically, the regular, finite and bounded region of thermoelastic shell is defined by

$$\varepsilon = \left| \theta^3 \right| / |R_{\min}| \ll 1. \tag{42}$$

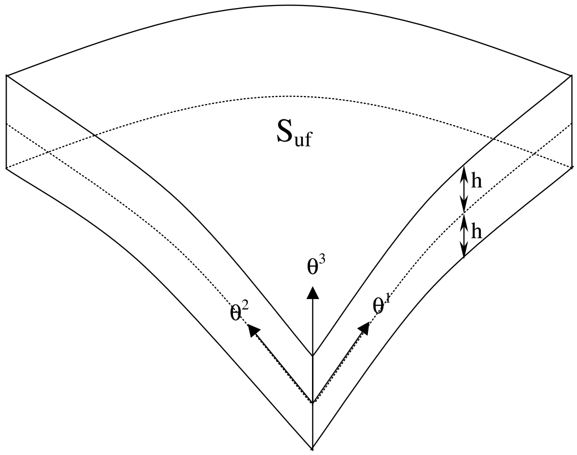

Fig. 1. An element of shell showing the reference surface $A(\theta^3 = 0)$ and upper and lower faces, $S_{\mathrm{uf}}(\theta^3 = \hbar)$ and $S_{\mathrm{lf}}(\theta^3 = -\hbar)$.

Here, $2\hbar$ is the uniform thickness of shell and $R_{\min}$ denotes the least principal radius of curvature of the reference surface A. This fundamental assumption allows to treat the shell region as a two-dimensional medium, and also, it is a sufficient restriction in order to ensure the existence of the shell tensor or shifters μ^α_β (Naghdi, 1963).

3.2. Preliminaries

Certain selected results which are intended merely to faciliate the development in the next sections are recorded from the differential geometry of a surface, a more elaborate account of which may be found in the treatises by Naghdi (1972) and Librescu (1975). To begin with, the components of metric tensor are given by

$$g_{\alpha\beta} = \mu^\lambda_\alpha \mu^\nu_\beta \alpha_{\lambda\nu}, \;\; g_{\alpha 3} = 0, \;\; g_{33} = 1; \;\; g^{\alpha\beta} = \left(\mu^{-1}\right)^\alpha_\lambda \left(\mu^{-1}\right)^\beta_\nu \alpha^{\lambda\nu}, \;\; g^{\alpha 3} = 0, \;\; g^{33} = 1 \tag{43a}$$

for the shell space and those by

$$a^{\alpha\beta} = g^{\alpha\beta}\left(\theta^\lambda, 0\right), \;\; \alpha_{\alpha 3} = 0, \;\; \alpha_{33} = 1; \;\; \alpha^{\alpha\beta} = g^{\alpha\beta}\left(\theta^\lambda, 0\right), \;\; \alpha^{\alpha 3} = 0, \;\; \alpha^{33} = 1 \tag{43b}$$

for the reference surface. In these equations, the components of shell tensor μ^α_β and those of its inverse which act as shifters between space and surface tensors are defined by

$$\mu^\alpha_\beta = \delta^\alpha_\beta - \theta^3 b^\alpha_\beta, \quad \mu^\alpha_\nu \left(\mu^{-1}\right)^\nu_\beta = \delta^\alpha_\beta \tag{44}$$

and the associated relationships are given by

$$\mu_{,3} = -\mu\left(\mu^{-1}\right)^\alpha_\beta b^\beta_\alpha, \quad \mu(\mu^{-1})^\alpha_\beta = \delta^\alpha_\beta + \theta^3(b^\alpha_\beta - b^\nu_\nu \delta^\alpha_\beta), \quad \mu = \mid \mu^\alpha_\beta \mid . \tag{45}$$

In the above equations, $\alpha_{\alpha\beta}$, $b_{\alpha\beta}$ and $c_{\alpha\beta}\left(= b^\nu_\alpha b_{\nu\beta}\right)$ denote, respectively, the first, second and third fundamental forms of the reference surface A and its mean and Gaussian curvatures are given by,

$$K_{\mathrm{m}} = \tfrac{1}{2} b^\alpha_\alpha, \qquad K_{\mathrm{g}} = \mid b^\alpha_\beta \mid \tag{46}$$

and hence Eq. (45) is expressed by

$$\mu = 1 - 2\theta^3 K_{\mathrm{m}} + \left(\theta^3\right)^2 K_{\mathrm{g}}. \tag{47}$$

By use of the shifters, the components of a vector field, (χ^i, χ_i) and $(\overline{\chi}^i, \overline{\chi}_i)$, which are referred to the base vectors of shell space V and those of reference surface A, respectively, are associated with one another in the form,

$$\chi_\alpha = \mu_\alpha^\beta \overline{\chi}_\beta, \quad \chi^\alpha = \left(\mu^{-1}\right)_\beta^\alpha \overline{\chi}^\beta, \quad \overline{\chi}_\alpha = \left(\mu^{-1}\right)_\alpha^\beta \chi_\beta, \quad \overline{\chi}^\alpha = \mu_\beta^\alpha \chi^\beta, \quad \chi^3 = \chi_3 = \overline{\chi}_3 = \overline{\chi}^3. \tag{48}$$

Furthermore, the relationships of the form,

$$\begin{aligned} &\chi_{\alpha;\beta} = \mu_\alpha^\nu (\overline{\chi}_{\nu:\beta} - b_{\nu\beta}\overline{\chi}^3), \quad \chi_{;\beta}^\alpha = \left(\mu^{-1}\right)_\nu^\alpha (\overline{\chi}_{:\beta}^\nu - b_\beta^\nu \overline{\chi}^3), \\ &\chi_{\alpha;3} = \mu_\alpha^\beta \overline{\chi}_{\beta,3}, \quad \chi_{3;\alpha} = \overline{\chi}_{3,\alpha} + b_\alpha^\beta \overline{\chi}_\beta, \\ &\chi_{;3}^\alpha = \left(\mu^{-1}\right)_\beta^\alpha \overline{\chi}_{,3}^\beta, \quad \chi_{;\alpha}^3 = \overline{\chi}_{,\alpha}^3 + b_{\alpha\beta}\overline{\chi}^\beta, \\ &\chi_{;3}^3 = \chi_{3;3} = \chi_{3,3} = \overline{\chi}_{3,3} = \overline{\chi}_{,3}^3 \end{aligned} \tag{49}$$

for the vector field and the identities of the form,

$$\begin{aligned} \mu\mu_\alpha^\nu \chi_{;\beta}^{\alpha\beta} &= (\mu\mu_\eta^\nu \chi^{\eta\beta})_{:\beta} - \mu\mu_\alpha^\nu \left(\mu^{-1}\right)_\lambda^\beta b_\beta^\lambda \chi^{\alpha 3} - \mu b_\beta^\nu \chi^{3\beta}, \\ \mu\chi_{;3}^{3\alpha} &= \left(\mu\chi^{3\alpha}\right)_{;\alpha} + \mu\mu_\alpha^\nu b_{\nu\beta}\chi^{\alpha\beta} - \mu\left(\mu^{-1}\right)_\nu^\alpha b_\alpha^\nu \chi^{33}, \\ \mu_\alpha^\beta \chi_{;3}^{\alpha 3} &= \left(\mu_\alpha^\beta \chi^{\alpha 3}\right)_{,3} \end{aligned} \tag{50a}$$

and

$$\mu\chi_{;\alpha}^\alpha = (\mu\chi^\alpha)_{:\alpha} - \mu\left(\mu^{-1}\right)_\beta^\alpha b_\alpha^\beta \chi^3 \tag{50b}$$

for a tensor field χ^{ij} are recorded. Here and henceforth, colons are used to denote covariant differentiation with respect to the indicated space coordinate by use of surface metrics and semicolons that by use of space metrics, and overbars indicate the quantities which are referred to the base vectors of reference surface. Besides, in the shell space $V + S$, the volume elements $\mathrm{d}V$ and the surface element $\mathrm{d}S$ on the faces and the area element $\mathrm{d}A$ on the reference surface are given by,

$$\mathrm{d}V = \sqrt{g}\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3 = \mathrm{d}S\,\mathrm{d}\theta^3 = \mu\mathrm{d}A\,\mathrm{d}\theta^3, \qquad \mathrm{d}A = \sqrt{\mathrm{a}}\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \qquad \alpha = |\mathrm{a}_{\alpha\beta}| \tag{51a}$$

and the element of line $\mathrm{d}c$ by

$$n_\alpha\,\mathrm{d}S = \mu\nu_\alpha \mathrm{d}c\,\mathrm{d}\theta^3 \tag{51b}$$

along the Jordan curve C.

3.3. Method of reduction

In analyzing the physical response of shells, the two-dimensional equations are almost always deduced from the three-dimensional fundamental equations of continuum by means of a method of reduction. Among the methods of reduction such as the direct method, the asymptotic method and Mindlin's method of reduction (see Mindlin (1968, 1989); Reddy and Robins (1994) and Noor (1994) for a review; also see Genevey (1997)), the latter is used herein in systematically deriving the hierarchic system of two-dimensional equations for the motions of a thermoelastic shell. Mindlin's method of reduction is entirely reliant on a variational averaging procedure together with an initially selected kinematic hypothesis. The selection of kinematic hypothesis is evidently due to the fact that the differentiation operation is usually simpler than the integration operation in deriving the lower-order equations of structures. The method is of wide use in establishing one- and two-dimensional equations of structures (see Mindlin (1968, 1989) and Dökmeci

(1994) for a review of the subject), and it allows to incorporate all the significant effects except the influence of the shell parameter Eq. (42).

3.4. Kinematics

In the shell space $V + S$, all the field variables together with their derivatives are taken to be exist, single-valued and piecewise continuous functions of the space coordinates θ^i and time t under the suitable regularity and smoothness assumptions for the region as well as the thermo-mechanical loadings of shell. Then, in accordance with the fundamental assumption Eq. (42), the shifted components of displacements, that is, those referred to the base vectors of reference surface, are represented by

$$\bar{u}_i(\theta^j, t) = \sum_{n=0}^{N} \alpha_n(\theta^3) u_i^{(n)}(\theta^\alpha, t). \tag{52}$$

Here, from the mathematical point of view, a separation of variables solution is sought for the three-dimensional fundamental equations of continuum. Thus, the vector functions $u_i^{(n)}$ are unknown a priori and independent functions of the displacement components of order (n) to be determined, and they are assumed to exist and to be a function of class C_{mn}. The coordinate (shape) functions α_n can be chosen to be any type of functions which should be complete, and they are taken, due to Weierstrass's theorem, in the form,

$$\alpha_n = \left(\theta^3\right)^n. \tag{53}$$

The series expansions (52) and (53) of displacement components have sufficient kinematic freedom to incorporate as many higher order mechanical and thermal effects as deemed desirable in any case under consideration. Besides, N denotes the order of approximation in the series expansions and $N = 1$ is the closest to the kinematics used in Love's first and second approximations in the classical theory of elastic shells (Naghdi, 1972).

It follows from the derivatives of displacement components (52) and (53) that all the field variables and, for instance, the strain components may be similarly expressed by a power series expansions of the thickness coordinate, namely

$$e_{ij} = \sum_{n=0}^{N=\infty} \left(\theta^3\right)^n e_{ij}^{(n)}(\theta^\alpha, t). \tag{54}$$

To obtain explicitly the variational strain components, consider the mechanical part of gradient equations in Eq. (41) as follows:

$$\delta L_{\mathrm{m}}\{t^{ij}\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z L_{ij}^{\mathrm{m}} \delta t^{ij} \mu \, \mathrm{d}\theta^3. \tag{55}$$

This variational equation can be expressed by

$$\begin{aligned} \delta L_{\mathrm{m}}\{t^{ij}\} = &\int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \left\{ e_{\alpha\beta} - \frac{1}{2}[\mu_\alpha^\nu(\bar{u}_{\nu:\beta} - b_{\nu\beta}\bar{u}_3) + \mu_\beta^\nu(\bar{u}_{\nu:\alpha} - b_{\nu\alpha}\bar{u}_3)] \right\} \delta t^{\alpha\beta} \mu \, \mathrm{d}\theta^3 \\ &+ \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \left\{ 2\left[e_{\alpha 3} - \frac{1}{2}(\mu_\alpha^\beta \bar{u}_{\beta,3} + \bar{u}_{3,\alpha} + b_\alpha^\beta \bar{u}_\beta) \right] \delta t^{\alpha 3} + (e_{33} - \bar{u}_{3,3}) \delta t^{33} \right\} \mu \, \mathrm{d}\theta^3 \end{aligned} \tag{56}$$

in terms of the shifted components (48) of displacement vector. Inserting Eqs. (52) and (53) into Eq. (56) and carrying out the integrations across the thickness of shell, one finally arrives at the variational equation of the form,

$$\delta L_{\rm m}\{T^{ij}_{(n)}\} = \int_T {\rm d}t \int_A \sum_{n=0}^{N} (e^{(n)}_{ij} - E^{(n)}_{ij})\delta T^{ij}_{(n)} {\rm d}A. \tag{57}$$

Here, the strain components of order (n) are defined by

$$\begin{aligned} E^{(n)}_{\alpha\beta} &= \tfrac{1}{2}\Big[\Big(u^{(n)}_{\alpha:\beta} + u^{(n)}_{\beta:\alpha} - 2b_{\alpha\beta}u^{(n)}_3\Big) - \Big(b^{\nu}_{\alpha}u^{(n-1)}_{\nu:\beta} + b^{\nu}_{\beta}u^{(n-1)}_{\nu:\alpha} - 2c_{\alpha\beta}u^{(n-1)}_3\Big)\Big], \\ E^{(n)}_{\alpha 3} &= \tfrac{1}{2}\Big[(n+1)u^{(n+1)}_{\alpha} + u^{(n)}_{3,\alpha} - (n-1)b^{\beta}_{\alpha}u^{(n)}_{\beta}\Big] \quad \text{in } AXT, \\ E^{(n)}_{33} &= (n+1)u^{(n+1)}_3. \end{aligned} \tag{58}$$

Also, in Eq. (57), the stress resultants of order (n) are introduced by

$$T^{ij}_{(n)} = t^{ij}_{(n)} - 2K_{\rm m}t^{ij}_{(n+1)} + K_{\rm g}t^{ij}_{(n+2)}, \tag{59a}$$

where

$$\left(t^{ij}_{(n)}, T^{ij}_{(n)}\right) = \int_Z t^{ij}(1,\mu)\left(\theta^3\right)^n {\rm d}\theta^3. \tag{59b}$$

Evidently, the strain components of order (n) and the stress resultants of order (n) are functions of the aerial coordinates (θ^{α}) and time (t), only.

3.5. Distributions of the thermal field

To be consistent with the series expansions of displacement components Eqs. (52) and (53), the temperature increment of a generic point in the shell space is expressed by

$$\Theta\left(\theta^i, t\right) = \sum_{n=0}^{M=\infty} \left(\theta^3\right)^n \Theta_{(n)}(\theta^{\alpha}, t) \tag{60}$$

and from the derivatives of this equation, the thermal field vector is given by

$$e_i\left(\theta^j, t\right) = \sum_{n=0}^{M} \left(\theta^3\right)^n e^{(n)}_i(\theta^{\alpha}, t) \tag{61}$$

in terms of the thickness coordinate. As in the above, substituting Eqs. (60) and (61) into the thermal part of gradient equations in Eq. (41), namely,

$$\delta L_{\rm th}\{h^i\} = \int_T {\rm d}t \int_A {\rm d}A \int_Z L^t_i \delta h^i \mu\, {\rm d}\theta^3 \tag{62}$$

and performing the integration across the thickness, one finds the distributions of the thermal field in variational form as follows:

$$\delta L_{\rm th}\{H^i_{(n)}\} = \int_T {\rm d}t \int_A \sum_{n=0}^{M} (E^{(n)}_i - e^{(n)}_i)\delta H^i_{(n)} {\rm d}A. \tag{63}$$

Here, the components of thermal field of order (n) by

$$E^{(n)}_{\alpha} = -\Theta^{(n)}_{,\alpha}, E^{(n)}_3 = -(n+1)\Theta^{(n+1)} \quad \text{in } AXT \tag{64}$$

and the heat flux resultants of order (n) by

$$H^i_{(n)} = h^i_{(n)} - 2K_{\rm m}h^i_{(n+1)} + K_{\rm g}h^i_{(n+2)}, \tag{65a}$$

where

$$(h^i_{(n)}, H^i_{(n)}) = \int_Z h^i(1,\mu)\left(\theta^3\right)^n \mathrm{d}\theta^3 \tag{65b}$$

are introduced as the functions of aerial coordinates and time, only.

4. Constitutive equations of thermoelastic shell

With the stress and heat flux resultants which are surface tensors of the functions of aerial coordinates θ^α and time t, attention is now confined to the constitutive relations of thermoelastic shell. The constitutive relations are derived by means of the variational principle (41), though they may be obtained by use of the principle of virtual work or by an energy expression, or via the direct integration of local constitutive relations. For a non-linear elastic material, an anisotropic material and a temperature-dependent material of the thermoelastic shell, the macroscopic constitutive equations are recorded below.

4.1. Non-linear constitutive relations

Preparatory to the derivation of macroscopic constitutive equations, recall the constitutive mechanical part of the variational principle (41) together with Eqs. (5) and (8), as follows:

$$\delta L_{\mathrm{mc}}\{e_{ij}\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \left[t^{ij} - \frac{1}{2}\left(\frac{\partial B}{\partial e_{ij}} + \frac{\partial B}{\partial e_{ji}}\right)\right]\delta e_{ij}\mu\,\mathrm{d}\theta^3. \tag{66}$$

Introducing Eq. (54) into this variational equation and then performing the integrations across the thickness and keeping in mind the symmetry of stress tensor and the stress resultants (59), one has

$$\delta L_{\mathrm{mc}}\left\{e^{(n)}_{ij}\right\} = \int_T \mathrm{d}t \int_A \sum_{n=0}^{N}(T^{ij}_{(n)} - T^{ij}_{(n)c})\delta e^{(n)}_{ij}\mathrm{d}A, \tag{67}$$

where

$$T^{ij}_{(n)c} = \frac{1}{2}\left(\frac{\partial \mathscr{B}}{\partial e^{(n)}_{ij}} + \frac{\partial \mathscr{B}}{\partial e^{(n)}_{ji}}\right). \tag{68}$$

Here, recalling the piecewise continuity of the single-valued field quantities, the thermoelastic potential $\mathscr{B}$ per unit area of the reference surface A of the form

$$\mathscr{B} = \int_Z B\mu\,\mathrm{d}\theta^3 \tag{69}$$

is defined.

Likewise, the constitutive thermal part of the variational principle (41) is written as

$$\delta L_{\mathrm{tc}}\{e_i, \Theta\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \left[\left(h^i - \frac{\partial F}{\partial e_i}\right)\delta e_i + \left(\eta + \frac{\partial B}{\partial \Theta}\right)\delta\eta\right]\mu\,\mathrm{d}\theta^3 \tag{70}$$

where Eqs. (6)–(8) are considered. Inserting Eqs. (60) and (61) into Eq. (70) and carrying out the integrations across the thickness, one obtains

$$\delta L_{\mathrm{tc}}\{e^{(n)}_i, \Theta_{(n)}\} = \int_T \mathrm{d}t \int_A \sum_{n=0}^{M}[(H^i_{(n)} - H^i_{(n)c})\delta e^{(n)}_i + \left(N_{(n)} - N_{(n)c}\right)\delta\eta_{(n)}]\mathrm{d}A, \tag{71}$$

where

$$H^i_{(n)c} = \frac{\partial \mathscr{F}}{\partial e_i^{(n)}}, \qquad N_{(n)c} = -\frac{\partial \mathscr{B}}{\partial \Theta_{(n)}} \tag{72}$$

and the dissipation function $\mathscr{F}$ per unit area of the reference surface in the form

$$\mathscr{F} = \int_Z F\mu\,\mathrm{d}\theta^3 \tag{73}$$

is defined. Besides, the entropy density of the form

$$\eta = \sum_{n=0}^{M} (\theta^3)^n \eta_{(n)}(\theta^\alpha, t) \tag{74}$$

and the resultants of entropy density by

$$N_{(n)} = \int_Z \eta(\theta^3)^n \mu\,\mathrm{d}\theta^3 \tag{75}$$

are introduced in accordance with the resultants of stress and heat flux (59) and (65).

4.2. Linear constitutive equations

Making use of the quadratic versions of the thermoelastic potential (9) and (10), the linear counterparts of the constitutive equations of thermoelastic shell (68), (72) and (75) are found to be

$$T^{ij}_{(n)c} = \sum_{m=0}^{N} \left(C^{ijkl}_{(m+n)} e^{(m)}_{kl} - \Omega^{ij}_{(m+n)} \Theta_{(m)} \right) \tag{76}$$

and

$$H^i_{(n)c} = \sum_{m=0}^{M} K^{ij}_{(m+n)} e^{(m)}_j - \tau_0 \dot{H}^i_{(n)}, \tag{77}$$

$$N_{(n)c} = \sum_{m=0}^{M} \left(A_{(m+n)} \Theta_{(m)} + \Omega^{ij}_{(m+n)} e^{(m)}_{ij} \right). \tag{78}$$

Here, the elastic stiffenesses of thermoelastic shell of the form

$$\left(C^{ijkl}_{(n)}, K^{ij}_{(n)}, \Omega^{ij}_{(n)}, A_{(n)} \right) = \int_Z \left(c^{ijkl}, k^{ij}, \lambda^{ij}, \alpha \right) (\theta^3)^n \mu\,\mathrm{d}\theta^3 \tag{79}$$

are given. If attention is restricted to a shell material whose mechanical properties are homogeneous in the thickness coordinate, the elastic stiffenesses may be expressed by

$$\left(C^{ijkl}_{(n)}, K^{ij}_{(n)}, \Omega^{ij}_{(n)}, A_{(n)} \right) = \left(c^{ijkl}, k^{ij}, \lambda^{ij}, \alpha \right) \mu_{(n)}, \tag{80}$$

where the shell tensor of order (n), namely

$$\mu_{(n)} = \int_Z \mu(\theta^3)^n \mathrm{d}\theta^3, \tag{81}$$

then with the aid of Eq. (47),

$$\mu_{(n)} = I_{(n)} - 2K_{\mathrm{m}} I_{(n+1)} + K_{\mathrm{g}} I_{(n+2)} \tag{82}$$

with the denotations by

$$\begin{aligned} I_{(n)} &= \int_Z (\theta^3)^n \mathrm{d}\theta^3, \\ I_{(n=2p)} &= 2\frac{h^n+1}{(n+1)}, \qquad I_{(n=2p+1)} = 0 \end{aligned} \tag{83}$$

is defined.

4.3. Temperature-dependent materials

In view of the temperature dependency of material (18), the constitutive equations of thermoelastic shell are obtained by the use of Eqs. (54), (60) and (61) and (80)–(82) in Eqs. (76)–(78) as follows:

$$\begin{aligned} T^{ij}_{(n)c} = \sum_{m=0}^{N} \Bigg\{ & \mu_{(m+n)} \left(c_0^{ijkl} e_{kl}^{(m)} - \lambda_0^{ij} \Theta_{(m)} \right) + \sum_{p=0}^{N} \Bigg[\mu_{(m+p+n)} \left(c_1^{ijkl} e_{kl}^{(m)} - \lambda_1^{ij} \Theta_{(m)} \right) \Theta_{(p)} \\ & + \sum_{q=0}^{N} \mu_{(m+p+q+n)} \left(c_2^{ijkl} e_{kl}^{(m)} - \lambda_2^{ij} \Theta_{(m)} \right) \Theta_{(p)} \Theta_{(q)} \Bigg] \Bigg\} \end{aligned} \tag{84}$$

and

$$\begin{aligned} H^{i}_{(n)c} &= \sum_{m=0}^{M} \Bigg\{ \mu_{(m+n)} k_0^{ij} e_j^{(m)} + \sum_{p=0}^{N} \Bigg[\mu_{(m+p+n)} k_1^{ij} e_j^{(m)} \Theta_{(p)} + \sum_{q=0}^{M} \mu_{(m+p+q+n)} k_2^{ij} e_j^{(m)} \Theta_{(p)} \Theta_{(q)} \Bigg] \Bigg\} - \tau_0 \dot{H}^i_{(n)}, \\ N_{(n)c} &= \sum_{m=0}^{M} \Bigg\{ \mu_{(m+n)} \left(\alpha_0 \Theta_{(m)} + \lambda_0^{ij} e_{ij}^{(m)} \right) + \sum_{p=0}^{N} \Bigg[\mu_{(m+p+n)} \left(\alpha_1 \Theta_{(m)} + \lambda_1^{ij} e_{ij}^{(m)} \right) \Theta_{(p)} \\ & \qquad + \sum_{q=0}^{N} \mu_{(m+p+q+n)} \left(\alpha_2 \Theta_{(m)} + \lambda_2^{ij} e_{ij}^{(m)} \right) \Theta_{(p)} \Theta_{(q)} \Bigg] \Bigg\} \end{aligned} \tag{85}$$

in terms of Eqs. (65), (69) and (75).

5. Thermoelastic shell equations

This section deals with the main topic of the paper, that is, a consistent and systematic derivation of the remaining equations of thermoelastic shell. The derivation rests entirely on (i) the fields of displacements and temperature increment chosen a priori, (ii) representing the two fields by the power series expansions in the thickness coordinate and (iii) employing an averaging procedure of variational type. The first of which is an important choice of the basic independent variables and makes the derivation comprehensive and tractable as already noted in the previous section. The second is almost compulsory in order to account satisfactorily for the high-frequency vibrations of thermoelastic shell. The averaging procedure is tacitly an implication of the fact that all the field variables are taken to be exist, continuous and not varied widely across the shell thickness. In the derivation, stresses or strains (Chien, 1944a,b; Simmonds, 1984; Pietraszkiewicz, 1998) and heat fluxes or thermal field can be alternatively chosen for the mechanical and thermal parts, respectively, as the basic variables. However, the choice of displacements and temperature increment represents a unification of the kinematics of classical shell and plate theories (Jemielita, 1990) and hence it seems to be more convenient for the derivation.

5.1. Equations of motion

To begin with, consider the variational stress equations of motion in Eq. (41), namely

$$\delta L_{\mathrm{mu}}\{u_i\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z L^i_{\mathrm{m}} \delta u_i \mu \, \mathrm{d}\theta^3. \tag{86}$$

With the aid of relationships (1a) and relationships (49), this equation may be written as

$$\delta L_{\mathrm{mu}}\{\overline{u}_i\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \left[(t^{\beta\alpha}_{;\beta} + t^{3\alpha}_{;3} + \rho f^\alpha)\mu^\nu_\alpha \delta\overline{u}_\nu + (t^{\alpha 3}_{;\alpha} + t^{33}_{;3} + \rho f^3)\delta\overline{u}_3 - \rho\ddot{\overline{u}}^i \delta\overline{u}_i \right] \mu \, \mathrm{d}\theta^3. \tag{87}$$

in terms of the shifted components of displacements. Following the use of identities (50) and hence expressing the covariant differentiations with respect to the metric tensor $\alpha_{\alpha\beta}$ in lieu of the metric tensor g_{ij}, Eq. (87) may be put in the form,

$$\begin{aligned} \delta L_{\mathrm{mu}}\{\overline{u}_i\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \Big\{ \Big[\left(\mu\mu^\alpha_\nu t^{\beta\nu}\right)_{:\beta} - \mu\mu^\alpha_\nu \left(\mu^{-1}\right)^\beta_\lambda b^\lambda_\beta t^{\nu 3} - \mu b^\alpha_\nu t^{\nu 3} + \mu\left(\mu^\alpha_\nu t^{\nu 3}\right)_{,3} + \rho(f^\alpha - \ddot{\overline{u}}^\alpha)\mu \Big] \delta\overline{u}_\alpha \\ + \Big[\left(\mu t^{3\alpha}\right)_{:\alpha} + \mu\mu^\nu_\alpha b_{\nu\beta} t^{\alpha\beta} - \mu\left(\mu^{-1}\right)^\alpha_\beta b^\beta_\alpha t^{33} + \mu t^{33}_{;3} + \rho(f^3 - \ddot{\overline{u}}^3) \Big] \delta\overline{u}_3 \Big\} \mathrm{d}\theta^3. \end{aligned} \tag{88}$$

Substituting Eq. (52) into this equation, recalling relations (45) and integrating with respect to the thickness coordinate, one finally arrives at the variational equation of shell motion as

$$\delta L_{\mathrm{mu}}\{u^{(n)}_i\} = \int_T \mathrm{d}t \int_A \sum_{n=0}^{N} \left(V^i_{(n)} - \rho \ddot{A}^i_{(n)} \right) \delta u^{(n)}_i \, \mathrm{d}A \tag{89}$$

with the denotations of the form,

$$\begin{aligned} V^\alpha_{(n)} &= \left(T^{\beta\alpha}_{(n)} - b^\alpha_\nu T^{\beta\nu}_{(n+1)} \right)_{:\beta} - b^\alpha_\beta T^{\beta 3}_{(n)} - n\left(T^{3\alpha}_{(n-1)} - b^\alpha_\beta T^{3\beta}_{(n)} \right) + S^\alpha_{(n)} + \rho\left(F^\alpha_{(n)} - b^\alpha_\beta F^\beta_{(n+1)} \right), \\ V^3_{(n)} &= T^{\alpha 3}_{(n):\alpha} + b_{\alpha\beta} T^{\alpha\beta}_{(n)} - c_{\alpha\beta} T^{\alpha\beta}_{(n+1)} - n T^{33}_{(n-1)} + S^3_{(n)} + \rho F^3_{(n)} \end{aligned} \tag{90}$$

in terms of the stress resultants (59). Also, in this equation, the acceleration resultants are introduced by

$$\ddot{A}^{(n)}_i = \sum_{m=0}^{N} \mu_{(n+m)} \ddot{u}^{(m)}_i, \tag{91}$$

the body force resultants by

$$F^i_{(n)} = \int_Z f^i \left(\theta^3\right)^n \mu \, \mathrm{d}\theta^3, \tag{92}$$

the surface loads by

$$\left(P^i_{(n)}, Q^i_{(n)} \right) = \mu\left(t^{3i} - \theta^3 b^\alpha_\beta t^{3\beta} \delta^i_\alpha \right) \left(\theta^3\right)^n \quad \text{at } \theta^3 = (-h, h), \tag{93a}$$

and the effective surface loads by

$$S^i_{(n)} = Q^i_{(n)} - P^i_{(n)}. \tag{93b}$$

5.2. Heat conduction equation

Analogously, and with the use of Eq. (2), the variational equation of heat conduction in Eq. (41), namely

$$\delta L_{\mathrm{tt}}\{\Theta\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \left(h^i_{;i} + \rho\not{h} + \Theta_0 \dot{\eta} \right) \mu \delta\Theta \mathrm{d}\theta^3 \tag{94}$$

is evaluated to obtain

$$\delta L_{\mathrm{tt}}\{\Theta\} = \int_T \mathrm{d}t \int_A \left[(\mu h^\alpha)_{:\alpha} + \left(\mu h^3\right)_{,3} + \rho\not{h} + \Theta_0 \dot{\eta} \right] \delta\Theta \mathrm{d}\theta^3, \tag{95}$$

where relationships (45) and (49) and indentities (50) are considered. Next, substitution of Eq. (60) into this equation and then integration with respect to the thickness coordinate results in

$$\delta L_{\mathrm{tt}}\left\{\Theta_{(n)}\right\} = \int_T \mathrm{d}t \int_A \sum_{n=0}^{N} V_{(n)} \delta\Theta_{(n)} \, \mathrm{d}A \tag{96}$$

with the denotations of the form

$$V_{(n)} = H^\alpha_{(n):\alpha} - nH^3_{(n-1)} + \rho F_{(n)} + \Theta_0 \dot{N}_{(n)} + H_{(n)} \tag{97}$$

in terms of the resultants of heat flux (65) and those of entropy density (75). In Eq. (97), the heat source resultants are introduced by

$$F_{(n)} = \int_Z \not{h} \left(\theta^3\right)^n \mu \, \mathrm{d}\theta^3, \tag{98}$$

the thermal loads by

$$\left(E_{(n)}, G_{(n)}\right) = \mu \not{h}^3 \left(\theta^3\right)^n \quad \text{at } \theta^3 = (-\not{h}, \not{h}) \tag{99a}$$

and hence,

$$H_{(n)} = G_{(n)} - E_{(n)}. \tag{99b}$$

5.3. Mechanical boundary conditions

The tractions are taken to be prescribed on the faces $S_{\mathrm{f}} (= S_{\mathrm{lf}} \cup S_{\mathrm{uf}})$ and on some part C_{t} of the edge boundary surface S_{e}, while the displacements are given on the remaining part C_{u} of the edge boundary surface. Accordingly, the mechanical boundary conditions in Eq. (41) are expressed in the form

$$\begin{aligned} \delta L_{*\mathrm{m}}\{u_i, t^{ij}\} = & \int_T \mathrm{d}t \int_{S_{\mathrm{f}}} \left(t^i_* - n_3 t^{3i}\right) \delta u_i \mathrm{d}S + \int_T \mathrm{d}t \int_{C_{\mathrm{t}}} \mathrm{d}c \int_Z \left(t^i_* - v_\alpha t^{\alpha i}\right) \mu \delta u_i \, \mathrm{d}\theta^3 \\ & + \int_T \mathrm{d}t \int_{C_{\mathrm{u}}} \mathrm{d}c \int_Z v_\alpha \left(u_i - u^*_i\right) \delta t^{\alpha i} \mu \, \mathrm{d}\theta^3, \end{aligned} \tag{100}$$

where C_{u} and C_{t} are the complementary parts of the Jordan curve C. By evaluating the surface integrals with the aid of Eqs. (48), (52) and (59), as before, one reads

$$\begin{aligned} \delta L_{*\mathrm{m}}\left\{u^{(n)}_i, T^{ij}_{(n)}\right\} = & \int_T \mathrm{d}t \int_{S_{\mathrm{uf}}} \sum_{n=0}^{N} \left(Q^i_{*(n)} - Q^{(i)}_{(n)}\right) \delta u^{(n)}_i \, \mathrm{d}A + \int_T \mathrm{d}t \int_{S_{\mathrm{lf}}} \sum_{n=0}^{N} \left(P^i_{*(n)} + P^{(i)}_{(n)}\right) \delta u^{(n)}_i \, \mathrm{d}A \\ & + \int_T \mathrm{d}t \int_{C_{\mathrm{t}}} \sum_{n=0}^{N} \left(V^i_{*(n)} - v_\alpha V^{\alpha i}_{(n)}\right) \delta u^{(n)}_i \mathrm{d}c + \int_T \mathrm{d}t \int_{C_{\mathrm{u}}} \sum_{n=0}^{N} v_\alpha \left(u^{(n)}_i - u^{*(n)}_i\right) \delta T^{\alpha i}_{(n)} \, \mathrm{d}c, \end{aligned} \tag{101}$$

where the denotations of the form

$$V^{\alpha\beta}_{(n)} = T^{\alpha\beta}_{(n)} - b^{\alpha}_{\nu} T^{\beta\nu}_{(n+1)}, \qquad V^{\alpha 3}_{(n)} = T^{\alpha 3}_{(n)} \tag{102}$$

and the traction resultants as follows:

$$\left(P^{i}_{*(n)}, Q^{i}_{*(n)}\right) = \mu\left(t^{i}_{*} - \theta^3 b^{\alpha}_{\beta} t^{\beta}_{*} \delta^{i}_{\alpha}\right) \quad \text{at } \theta^3 = (-h, h), \tag{103a}$$

$$V^{i}_{*(n)} = T^{i}_{*(n)} - b^{\alpha}_{\beta} T^{\beta}_{*(n)} \delta^{i}_{\alpha}, \tag{103b}$$

where

$$T^{i}_{*(n)} = \int_Z t^{i}_{*} \left(\theta^3\right)^n \mu \, \mathrm{d}\theta^3 \tag{103c}$$

are defined.

5.4. Thermal boundary conditions

To prescribe the temperature increment is a difficult one to materialize physically and hence only the boundary conditions involving with the heat fluxes are considered. Thus, the thermal boundary conditions in Eq. (41) read

$$\delta L_{*t}\{\Theta\} = \int_T \mathrm{d}t \int_{S_\mathrm{f}} \left(h_* - n_3 h^3\right) \delta\Theta \, \mathrm{d}S + \int_T \mathrm{d}t \oint_C \mathrm{d}c \int_Z \left(h_* - \nu_\alpha h^\alpha\right) \mu \delta\Theta \, \mathrm{d}\theta^3. \tag{104}$$

Paralleling to the evaluation of Eq. (100) and using Eqs. (60), (65a) and (75), this equation takes the form,

$$\begin{aligned} \delta L_{*t}\left\{\Theta_{(n)}\right\} = {} & \int_T \mathrm{d}t \int_{S_\mathrm{uf}} \sum_{n=0}^{N} \left(G^{*}_{(n)} - G_{(n)}\right) \delta\Theta_{(n)} \, \mathrm{d}A + \int_T \mathrm{d}t \int_{S_\mathrm{lf}} \sum_{n=0}^{N} \left(E^{*}_{(n)} + E_{(n)}\right) \delta\Theta_{(n)} \, \mathrm{d}A \\ & + \int_T \mathrm{d}t \oint_C \sum_{n=0}^{N} \left(H^{(n)}_{*} - \nu_\alpha H^{\alpha}_{(n)}\right) \delta\Theta_{(n)} \, \mathrm{d}c, \end{aligned} \tag{105}$$

where

$$\left(E^{(n)}_{*}, G^{(n)}_{*}\right) = \mu h_* \left(\theta^3\right)^n \quad \text{at } \theta^3 = (-h, h), \tag{106}$$

$$H_{*(n)} = \int_Z \mu \left(\theta^3\right)^n h_* \, \mathrm{d}\theta^3 \tag{107}$$

are the heat flux resultants which are prescribed at the faces and on the edge boundary surface, respectively.

5.5. Hierarchical initial conditions

In view of Eqs. (23)–(25), a set of initial conditions based on the series expansions of displacements (52) and temperature increment (60) may be expressed by

$$u^{(n)}_{i}(\theta^\alpha, t_0) - \alpha^{*(n)}_{i}(\theta^\alpha) = 0, \qquad \dot{u}^{(n)}_{i}(\theta^\alpha, t_0) - \beta^{*(n)}_{i}(\theta^\alpha) = 0 \quad \text{on } A(t_0) \tag{108}$$

and

$$\Theta_{(n)}(\theta^\alpha, t_0) - \gamma^{*}_{(n)}(\theta^\alpha) = 0 \quad \text{on } A(t_0), \tag{109}$$

where $\alpha^{*(n)}_{i}$, $\beta^{*(n)}_{i}$ and $\gamma^{*}_{(n)}$ stand for the series representations of the prescribed functions α^{*}_{i}, β^{*}_{i} and γ_* in terms of the thickness coordinate.

5.6. Thermoelastic shell equations

Now, substituting Eqs. (57), (67), (89) and (101) into the mechanical part, and Eqs. (63), (71), (96) and (105) into the thermal part, of the unified variational principle (41), one has the hierarchic system of two-dimensional, approximate equations in variational form for the high-frequency, extensional, thickness-shear, flexural and torsional as well as coupled motions of thermoelastic shell of uniform thickness in the form,

$$\delta L_S^{(n)}\{\wedge_S^{(n)}\} = \delta L_{S_\mathrm{m}}^{(n)}\{\wedge_{S_\mathrm{m}}^{(n)}\} + \delta L_{S_\mathrm{t}}^{(n)}\{\wedge_{S_\mathrm{t}}^{(n)}\} = 0, \tag{110}$$

where

$$\begin{aligned}\delta L_{S_\mathrm{m}}^{(n)}\{\wedge_{S_\mathrm{m}}^{(n)}\} &= \delta L_\mathrm{m}\left\{T_{(n)}^{ij}\right\} + \delta L_\mathrm{mc}\left\{e_{ij}^{(n)}\right\} + \delta L_\mathrm{mu}\left\{u_i^{(n)}\right\} + \delta L_{*\mathrm{m}}\left\{u_i^{(n)}, T_{ij}^{(n)}\right\},\\ \delta L_{S_\mathrm{t}}^{(n)}\{\wedge_{S_\mathrm{t}}^{(n)}\} &= \delta L_\mathrm{th}\left\{H_{(n)}^i\right\} + \delta L_\mathrm{te}\left\{e_i^{(n)}, \eta_{(n)}\right\} + \delta L_\mathrm{tt}\left\{\Theta_{(n)}\right\} + \delta L_{*\mathrm{t}}\left\{\Theta_{(n)}\right\}\end{aligned} \tag{111}$$

with the admissible states of the form

$$\begin{aligned}&\wedge_S^{(n)} = \wedge_{S_\mathrm{m}}^{(n)} \cup \wedge_{S_\mathrm{t}}^{(n)},\\ &\wedge_{S_\mathrm{m}}^{(n)} = \left\{u_i^{(n)}, e_{ij}^{(n)}, T_{ij}^{(n)}\right\}, \qquad \wedge_{S_\mathrm{t}}^{(n)} = \left(\Theta_{(n)}, e_i^{(n)}, H_i^{(n)}, \eta_{(n)}\right)\end{aligned} \tag{112}$$

in terms of the displacement components (52), strain components (57) and stress resultants (59), of order n, and the temperature increment (60), thermal field components (61), heat flux resultants (65) and entropy resultants (74), of order n. Setting the variational Eq. (110) equal to zero for the arbitrary and independent variations of the admissible state (112), the hierarchic system of equations in differential form is expressed by

$$e_{ij}^{(n)} - E_{ij}^{(n)} = 0 \quad \text{on } AXT, \tag{113}$$

$$T_{(n)}^{ij} - T_{(n)c}^{ij} = 0 \quad \text{on } AXT, \tag{114}$$

$$V_{(n)}^i - \rho\ddot{A}_{(n)}^i = 0 \quad \text{on } AXT \tag{115}$$

in terms of Eqs. (58), (76) and (90), the boundary conditions by

$$\begin{aligned}&Q_{*(n)}^i - Q_{(n)}^i = 0 \quad \text{on } S_\mathrm{uf}XT,\ P_{*(n)}^i + P_{(n)}^i = 0 \quad \text{on } S_\mathrm{lf}XT,\\ &V_{*(n)}^i - v_\alpha V_{(n)}^{\alpha i} = 0 \quad \text{on } C_\mathrm{t}XT,\ u_i^{(n)} - u_i^{*(n)} = 0 \quad \text{on } C_\mathrm{u}XT\end{aligned} \tag{116}$$

in terms of Eq. (103), the initial conditions (108), the series expansions of displacement components (52) and the symmetry of stress tensor which is based on (1b), namely

$$\varepsilon_{ijk} T_{(n)}^{jk} = 0 \quad \text{on } AXT \tag{117}$$

for the mechanical part, and

$$e_i^{(n)} - E_i^{(n)} = 0 \quad \text{on } AXT \tag{118}$$

$$H_{(n)}^i - \mathrm{H}_{(n)c}^i = 0, \qquad N_{(n)} - N_{(n)c} = 0 \quad \text{on } AXT \tag{119}$$

$$V_{(n)} = 0 \quad \text{on } AXT \tag{120}$$

in terms of Eqs. (64), (77), (78) and (97), the boundary conditions by

$$G^{*}_{(n)} - G_{(n)} = 0 \quad \text{on } S_{\text{uf}}XT, \qquad E^{*}_{(n)} - E_{(n)} = 0 \quad \text{on } S_{\text{lf}}XT,$$
$$H^{(n)}_{*} - v_{\alpha} H^{\alpha}_{(n)} = 0 \quad \text{on } CXT \tag{121}$$

in terms of Eqs. (106) and (107), the initial conditions (109) and the series expansion of temperature increment (60) for the thermal part, of thermoelastic shell.

6. Special cases: uniqueness of solutions

Thus far, a hierarchic system of two-dimensional, non-isothermal, approximate equations is consistently derived in order to predict the high frequency motions of a thermoelastic shell having temperature-dependent material. The hierarchic system of shear-deformable, thermoelastic shell equations is formulated in invariant, both differential and variational forms, and hence it is reducible to any case of interest, involving special geometry, material, motions and alike. Some of special cases are recorded in this section and also, a theorem of uniqueness is devised in solutions of the hierarchic system of fully linearized shell equations.

On shell geometry: Due to its invariant feature, the hierarchic system of two-dimensional equations can be readily expressed in a particular system of coordinates most suitable to the geometrical configuration of thermoelastic shell. Besides, in the case of a shallow shell, the shell tensor or the shifters may be appropriately expressed by

$$\mu^{\alpha}_{\beta} \cong \delta^{\alpha}_{\beta}, \quad | \mu^{\alpha}_{\beta} | = \mu \cong 1, \quad \mu_{(n)} \cong I_{(n)} \tag{122a}$$

in Eqs. (45) and (82), and by

$$\chi_{\alpha;\beta} = \overline{\chi}_{\alpha:\beta} - b_{\alpha\beta}\overline{\chi}^{3}, \qquad \chi_{\alpha;3} = \overline{\chi}_{\alpha,3}, \qquad \chi_{3;\alpha} = \overline{\chi}_{3,\alpha} + b_{\alpha\beta}\overline{\chi}^{\beta} \tag{122b}$$

in Eq. (49). In addition, in the absence of curvature effects, the shell tensor is simply reduced to the Kronecker deltas and hence the hierarchic system of two-dimensional equations is given by the divergence equations (89) and (96) together with the relations of the form,

$$V^{i}_{(n)} = T^{\alpha i}_{(n):\alpha} - nT^{3i}_{(n-1)} + S^{i}_{(n)} + \rho F^{i}_{(n)},$$
$$S^{i}_{(n)} = h^{n}\left[t^{3i}\left(\theta^{\alpha}, \theta^{3} = \hbar, t\right) - t^{3i}\left(\theta^{\alpha}, \theta^{3} = -\hbar, t\right)\right] \tag{123}$$

the gradient equations (57) and (63) together with the relations of the form

$$E^{(n)}_{\alpha\beta} = \tfrac{1}{2}\left(u^{(n)}_{\alpha:\beta} + u^{(n)}_{\beta:\alpha}\right), \qquad E^{(n)}_{\alpha 3} = \tfrac{1}{2}\left[(n+1)u^{(n+1)}_{\alpha} + u^{(n)}_{3,\alpha}\right], \qquad E_{33} = (n+1)u^{(n+1)}_{3} \tag{124}$$

the constitutive relations (67) and (71), the boundary conditions (101) and (105) together with the relations of the form

$$V^{\alpha i}_{(n)} = T^{\alpha i}_{(n)}, \left(P^{i}_{*(n)}, Q^{i}_{*(n)}\right) = \hbar^{n}\left[t^{i}_{*}\left(\theta^{\alpha}, \theta^{3} = \hbar, t\right), t^{i}_{*}\left(\theta^{\alpha}, \theta^{3} = -\hbar, t\right)\right], \tag{125}$$

the initial conditions (109) and the series expansions (52) and (60) for the high-frequency motions of thermoelastic plate having temperature-dependent material.

On the kinematics: In deriving the hierarchic system of thermoelastic shell equations, the kinematic assumptions (52) and (60), that is, the series expansions of displacement components and temperature increment are chosen as a basis at the outset. By truncating the series expansions for $N = 2$, one recovers the classical equations of thermoelastic thin shells in the sense of Mindlin (1968) and also those of elastic shells within the frame of Love's first approximation [i.e., $u^{(1)}_{\alpha} = -(u^{(0)}_{3,\alpha} + b^{\beta}_{\alpha}u^{(0)}_{\beta}), u^{(1)}_{3} = 0, \Theta_{(n)} = 0$] and Love's second approximation ($\overline{u}_{\alpha} = \overline{u}^{(0)}_{\alpha} + \theta^{3}u^{(1)}_{\alpha}, \overline{u}_{3} = u^{(0)}_{3}$ and $\Theta_{(n)} = 0$). Also, the aforementioned equations of thermoelastic plate are agree with those given by Mindlin (1974) where the effect of second sound is discarded but the piezoelectric effect is included, and, of course, with Lagrange's or Mindlin's equations of elastic plates for the case when the high order mechanical terms for $N \geqslant 2$ and all the thermal terms are abrogated, and also, with Mindlin's plate equations for thermoelastic vibrations of temperature-dependent materials (Altay and Dökmeci, 1997).

On the material: Various types of material specializations such as the isotropy of shell material or the elastic symmetry of shell material with respect to the reference surface (Green and Zerna, 1954) can be readily considered in the constitutive relations of thermoelastic shell with or without temperature-dependency of material, that is, Eqs. (84) and (85) or Eqs. (76)–(78). By discarding the effect of second sound and the temperature-dependency of material as well as all the terms involving with the temperature variations, one recovers the isothermal equations of elastic shells reported by Dökmeci (1973, 1974) who considered the effect of piezoelectricity and those developed by Yokoo and Matsunaga (1974), Librescu (1975, 1987) and Brull and Librescu (1982) who took account of the effect of geometrical non-linearity as well. A detailed study of special cases and certain cases with numerical applications will be reported later.

6.1. Uniqueness of solutions

Paralleling to the uniqueness theorems for the three-dimensional linear theory of elastostatics (Kirchhoff, 1859) and elastodynamics (Neumann, 1885) by the method of energy arguments, the uniqueness theorems are devised in solutions of the two-dimensional, linear theory of non-polar and polar, elastic and thermoelastic shells and plates (Green and Naghdi, 1971; Naghdi and Trapp, 1972; Naghdi, 1972; Dökmeci, 1973, 1978, 1994; Rubin, 1986; Altay and Dökmeci, 1997) at the low-frequency motions. In an analogous way to Neumann's theorem in elastodynamics and Weiner's theorem in thermoelastodynamics, the uniqueness is examined in solutions of the high-frequency equations of piezoelectric and thermopiezoelectric plates (Mindlin, 1968, 1974; Tiersten, 1969) and crystal surfaces (Dökmeci, 1974). Now, within the frame of the coupled theory of thermoelasticity with second sound, a theorem of uniqueness is proved in solutions of the initial, mixed boundary value problems characterized by the hierarchic system of linearized equations of thermoelastic shell of uniform thickness.

Theorem . *With reference to the θ^i-system of geodesic normal coordinates in the Euclidean space Ξ, given a regular region $V + S$ of thermoelastic shell, with its boundary surface $S(= S_{\mathrm{e}} \cup S_{\mathrm{lf}} \cup S_{\mathrm{uf}})$, closure $\overline{V}(= V \cup S)$ and midsurface A. At the time interval $T = [t_0, t_1)$, under a prescribed initial data, the region of thermoelastic shell is set in motion and this motion is maintained by application of assigned surface traction and heat supply and by application of prescribed velocity, displacements and temperature fields over appropriate portions of the boundary surface S. Now, let*

$$\wedge_S = \wedge_{S_{\mathrm{m}}} \cup \wedge_{S_{\mathrm{t}}}$$

with

$$\wedge_{S_{\mathrm{m}}} = \left\{ u_i^{(n)} \in C_{12}, e_{ij}^{(n)} \in C_{00}, T_{ij}^{(n)} \in C_{10} \right\},$$
$$\wedge_{S_{\mathrm{t}}} = \left\{ \Theta_{(n)} \in C_{00}, e_i^{(n)} \in C_{00}, H^i_{(n)} \in C_{11}, N_{(n)} \in C_{01} \right\} \quad \text{on } AXT$$

be an admissible state of single-valued functions which satisfies the hierarchic linear system of two-dimensional divergence equations (115) *and* (120), *gradient equations* (113) *and* (118), *constitutive relations* (114) *and* (119), *boundary conditions* (116) *and* (121), *initial conditions* (108) *and* (109) *and the symmetry of stress resultants* (117). *Also, let the mass density ρ and the specific heat C_v are positive everywhere on A and the symmetry relations* (14) *hold. Then, there exists at most one admissible state* (126) *which satisfies the aforementioned equations of thermoelastic shell.*

To prove the theorem of uniqueness, one follows the usual lines by considering two admissible states $\wedge_S^{(\alpha)}$ of the 23 $N = M$ dependent variables $(u_i^{(n)}, e_{ij}^{(n)}, T_{ij}^{(n)}; \Theta_{(n)}, e_i^{(n)}, H_i^{(n)}, N_{(n)})$, initially zero, identified by prime and double primes and their difference by

$$\wedge_S = \wedge_S' - \wedge_S'' \tag{126a}$$

with

$$u_i^{(n)} = u_i'^{(n)} - u_i''^{(n)}, e_{ij}^{(n)} = e_{ij}'^{(n)} - e_{ij}''^{(n)}, \ldots,$$
$$\Theta_{(n)} = \Theta_{(n)}' - \Theta_{(n)}'', \ldots, N_{(n)} = N_{(n)}' - N_{(n)}'', \quad n = 1, 2, \ldots, N \tag{126b}$$

in which each state comprises a solution of the twenty-three N equations, that is, that of the hierarchic system of non-isothermal linear equations of thermoelastic shell with no singularities of any type. In view of the linearity of the hierarchic system of shell equations, the difference state $\wedge_S$ of Eq. (126a) is a solution as well. Thus, one reads, in terms of the difference solution, the homogeneous divergence equations of the form,

$$I_S = I_{S_\mathrm{m}} + I_{S_\mathrm{t}} = 0, \tag{127}$$

where the mechanical divergence equations of the form

$$I_{S_\mathrm{m}} = \sum_{n=0}^{N} \int_A \left[V_{(n)}^i - \rho\left(F_{(n)}^i + \ddot{A}_{(n)}^i \right) \right] \dot{u}_i^{(n)} \, \mathrm{d}A \tag{128}$$

in terms of Eq. (115) and the thermal divergence equations of the form

$$I_{S_\mathrm{t}} = \sum_{n=0}^{N} \int_A \Theta_0^{-1} \left(F_{(n)} - V_{(n)} \right) \Theta_{(n)} \, \mathrm{d}A \tag{129}$$

in terms of Eq. (120) are employed.

Before proceeding further, the kinetic energy density is recalled by

$$K = \tfrac{1}{2} \rho \dot{u}^i \dot{u}_i, \tag{130}$$

the purely mechanical energy density by

$$U = \tfrac{1}{2} c^{ijkl} e_{ij} e_{kl}, \tag{131}$$

the purely thermal energy by

$$T = \tfrac{1}{2} \alpha \Theta^2, \tag{132}$$

and the second laws of thermodynamics (Fox, 1969; Naghdi and Trapp, 1972) by

$$Q = h^i e_i \geqslant 0. \tag{133}$$

In addition to Eq. (133), the energy densities (130)–(132) are positive-definite, by definition, and initially zero due to the initial conditions (108) and (109); so that the quantities K, U, T and Q have the same properties for the difference state (126). Thus, integrating Eqs. (130)–(132), one has the positive-definite quantities of the form,

$$(K_S, U_S, T_S) = \int_V (K, U, T) \mathrm{d}V \tag{134}$$

and

$$Q_S = \int_V Q\,\mathrm{d}V \tag{135}$$

for the shell region.

Next, taking time differentiation of Eq. (130) and integrating over the volume of thermoelastic shell, one readily obtains the rate of the kinetic energy (134) in the form,

$$\dot{K}_S = \int_A \mathrm{d}A \int_Z \rho\ddot{\bar{u}}^i\dot{\bar{u}}_i\mu\,\mathrm{d}\theta^3, \tag{136}$$

in terms of the shifted components of displacements (48). Substituting Eq. (52) into this equation and then performing the integration with respect to the thickness coordinate, one reads the rate of the kinetic energy as

$$\dot{K}_S = \sum_{n=0}^{N}\int_A \rho\ddot{A}^i_{(n)}\dot{u}_i^{(n)}\,\mathrm{d}A, \tag{137}$$

in terms of the acceleration resultants (91). The shell tensor (81) is employed in the integration of Eq. (136). In a similar manner, the rate of the purely mechanical energy (131) reads

$$\dot{U}_S = \int_A \mathrm{d}A \int_Z \left(t^{ij} + \lambda^{ij}\Theta\right)\dot{e}_{ij}\mu\,\mathrm{d}\theta^3, \tag{138}$$

where the symmetry of stress tensor (1b), the constitutive relations (11) and the symmetry relations (14) are considered. With the use of strain distributions (54) and (113) and the series of temperature increment (60), the rate of energy (138) results in

$$\dot{U}_S = \dot{U}_{S_\mathrm{m}} + \dot{U}_{S_\mathrm{t}} \tag{139}$$

with

$$\begin{aligned}\dot{U}_{S_\mathrm{m}} = \sum_{n=0}^{N}\int_A \Big\{ & T^{\alpha\beta}_{(n)}\left(\dot{u}^{(n)}_{\alpha:\beta} - b_{\alpha\beta}\dot{u}^{(n)}_3 - b^{\nu}_{\alpha}\dot{u}^{(n-1)}_{\nu:\beta} - c_{\alpha\beta}\dot{u}^{(n-1)}_3\right) + (n+1)T^{33}_{(n)}\dot{u}^{(n+1)}_3 \\ & + T^{\alpha 3}_{(n)}\left[(n+1)\dot{u}_\alpha + \dot{u}_{3,\alpha} - (n-1)b^{\beta}_{\alpha}\dot{u}^{(n)}_\beta\right]\Big\}\mathrm{d}A\end{aligned} \tag{140}$$

and

$$\dot{U}_{S_\mathrm{t}} = \sum_{n=0}^{N}\int_A \lambda^{ij}O_{(n)}\dot{e}^{(n)}_{ij}\,\mathrm{d}A \tag{141}$$

in terms of the stress resultants (59) and the temperature resultants of order (n) as

$$O_{(n)} = \sum_{m=0}^{N}\mu_{(m+n)}\Theta_{(m)}. \tag{142}$$

Likewise, the rate of the purely thermal energy (132) is stated by

$$\dot{T}_S = \int_A \mathrm{d}A \int_Z \alpha\dot{\Theta}\Theta\mu\,\mathrm{d}\theta^3 \tag{143}$$

which, after inserting Eq. (60) and integrating across the shell thickness and also taking Eq. (142) into account, becomes

$$\dot{T}_S = \sum_{n=0}^{N} \int_A \alpha \dot{O}_{(n)} \Theta_{(n)}. \tag{144}$$

Furthermore, from Eqs. (133) and (135) and by the use of Eqs. (61) and (64), analogously to Eq. (144), one obtains

$$Q_S = -\sum_{n=0}^{N} \int_A \left[H^{\alpha}_{(n)} \Theta^{(n)}_{,\alpha} + (n+1) H^{3}_{(n)} \Theta_{(n+1)} \right] \mathrm{d}A \geqslant 0 \tag{145}$$

in terms of Eq. (65).

Now, by substituting Eq. (90) into Eq. (128), one reads

$$I_{S_\mathrm{m}} = \sum_{n=0}^{N} \int_A \left\{ \left[\left(T^{\beta\alpha}_{(n)} - b^{\alpha}_{\nu} T^{\nu\beta}_{(n+1)} \right)_{:\beta} - b^{\alpha}_{\beta} T^{\beta 3}_{(n)} - n \left(T^{3\alpha}_{(n-1)} - b^{\alpha}_{\beta} T^{\beta 3}_{(n)} + S^{\alpha}_{(n)} \right) \right] \dot{u}^{(n)}_{\alpha} + \left(T^{\alpha 3}_{(n):\alpha} + b_{\alpha\beta} T^{\alpha\beta}_{(n)} - c_{\alpha\beta} T^{\alpha\beta}_{(n+1)} - n T^{33}_{(n-1)} + S^{3}_{(n)} \right) \dot{u}^{(n)}_{3} - \rho \ddot{A}^{i}_{(n)} \dot{u}^{(n)}_{i} \right\} \mathrm{d}A. \tag{146}$$

After applying the divergence theorem, this equation may be expressed by

$$I_{S_\mathrm{m}} = \sum_{n=0}^{N} \left\{ \int_A \left[- \left(T^{\beta\alpha}_{(n)} - b^{\alpha}_{\nu} T^{\nu\beta}_{(n+1)} \right) \dot{u}^{(n)}_{\alpha:\beta} - n T^{3\alpha}_{(n-1)} \dot{u}^{(n)}_{\alpha} + (n-1) b^{\alpha}_{\beta} T^{\beta 3}_{(n)} \dot{u}^{(n)}_{\alpha} - T^{\alpha 3}_{(n)} \dot{u}^{(n)}_{3,\alpha} + \left(b_{\alpha\beta} T^{\alpha\beta}_{(n)} - c_{\alpha\beta} T^{\alpha\beta}_{(n+1)} \right) \dot{u}^{(n)}_{3} + S^{i}_{(n)} \dot{u}^{(n)}_{i} - \rho \ddot{A}^{i}_{(n)} \dot{u}^{(n)}_{i} \right] \mathrm{d}A + \oint_C \nu_\beta \left[\left(T^{\beta\alpha}_{(n)} - b^{\alpha}_{\nu} T^{\beta\nu}_{(n+1)} \right) \dot{u}^{(n)}_{\alpha} + T^{\beta 3}_{(n)} \dot{u}^{(n)}_{3} \right] \nu_\beta \, \mathrm{d}c \right\}. \tag{147}$$

By comparing Eqs. (137) and (140) with Eq. (147), one has

$$I_{S_\mathrm{m}} = -\dot{K}_S - \dot{U}_{S_\mathrm{m}} + \chi_\mathrm{m}, \tag{148}$$

where

$$\chi_\mathrm{m} = \sum_{n=0}^{N} \left[\int_A S^{i}_{(n)} \dot{u}^{(n)}_{i} \, \mathrm{d}A + \oint_C \nu_\beta V^{\beta i}_{(n)} \dot{u}^{(n)}_{i} \mu \, \mathrm{d}c \right] \tag{149}$$

in terms of denotations (102). Likewise, putting Eq. (97) into Eq. (129), one writes

$$I_{S_\mathrm{t}} = \sum_{n=0}^{N} \int_A \left[\Theta_0^{-1} \left(-H^{\alpha}_{(n):\alpha} + n H^{3}_{(n-1)} \right) - \dot{N}_{(n)} - \Theta_0^{-1} H_{(n)} \right] \Theta_{(n)} \, \mathrm{d}A. \tag{150}$$

By the divergence theorem and use of Eqs. (78)–(80) this equation takes the form

$$I_{S_\mathrm{t}} = \sum_{n=0}^{N} \left\{ \int_A \left[\Theta_0^{-1} \left(H^{\alpha}_{(n)} \Theta^{(n)}_{,\alpha} + n H^{3}_{(n-1)} \Theta_{(n)} - H_{(n)} \Theta_{(n)} \right) - \alpha \dot{O}_{(n)} \Theta_{(n)} - \lambda^{ij} O_{(n)} \dot{e}^{(n)}_{ij} \right] \mathrm{d}A - \oint_C \Theta_0^{-1} \nu_\alpha H^{\alpha}_{(n)} \Theta_{(n)} \mu \, \mathrm{d}c \right\}. \tag{151}$$

This, comparing with Eqs. (141), (144) and (145), gives the result of the form

$$I_{S_t} = -\dot{T}_S - \dot{U}_{S_t} - Q_S + \chi_t. \tag{152}$$

In this equation,

$$\chi_t = \sum_{n=0}^{N} -\Theta_0^{-1}\left(\int_A H_{(n)}\Theta_{(n)}\,\mathrm{d}A + \oint_C v_\alpha H^\alpha_{(n)}\Theta_{(n)}\mu\,\mathrm{d}c\right) \tag{153}$$

is defined.

As a last step, assembling all the results of Eqs. (148) and (152) in (127), the latter reads

$$I_S = -\dot{K}_S - \dot{U}_S - \dot{T}_S - Q_S + \chi_m + \chi_t = 0. \tag{154}$$

In this equation, the last two terms (χ_m, χ_t) vanish due to the mechanical and thermal boundary conditions (116) and (121) and the initial conditions (108) and (109). Also, the conditions sufficient to make the two terms zero are: specification of one member of each of the (n)-products of displacement and face traction components as well as heat flux and temperature increment at each point of the interior of the thermoelastic shell, and also specification of one member of each of the (n)-products of edge-displacement and edge-traction components as well as edge-temperature increment and edge-heat flux at each point of the Jordan curve C of the thermoelastic shell. In view of these sufficient boundary and initial conditions and condition (145), Eq. (154) is expressed by

$$\dot{K}_S + \dot{U}_S + \dot{T}_S = -Q_S \leqslant 0. \tag{155}$$

Integration of this equation over the time interval T results in

$$K_S(t_1) + U_S(t_1) + T_S(t_1) \leqslant K_S(t_0) + U_S(t_0) + T_S(t_0). \tag{156}$$

By this result one finally has

$$K_S(t_1) = K_S(t_0) = U_S(t_1) = U_S(t_0) = T_S(t_1) = T_S(t_0) = 0 \tag{157}$$

due to the positive-definite properties of K, U and T mentioned above. Thus, one concludes that the difference set of solutions (126) is identically zero throughout the region of thermoelastic shell under the usual continuity conditions, the symmetry of stress tensor (1b) and the positive-definiteness of material elasticities.

7. Conclusions

In relation to high-frequency vibrations of temperature-dependent elastic materials, a hierarchic system of shear deformable shell equations was rigorously deduced from the three-dimensional fundamental equations of thermoelasticity with second sound. First, Hamilton's principle was stated for a thermoelastic medium and then a unified, differential type of variational principles was proposed which leads to the fundamental equations of thermoelasticity, as its Euler–Lagrange equations. Next, by use of the unified variational principle together with a priori chosen fields of displacements and temperature increment, a consistent reduction of the fundamental equations was accomplished to the approximate equations of thermoelastic shell of uniform thickness under thermo-mechanical loading. All the mechanical and thermal effects of higher orders as well as the quadratic temperature-dependency of elastic material with second sound were taken into account as deemed desirable in any case under consideration. Due to its invariant nature, the hierarchic system of equations can be readily expressible in any particular system of coordinates most suitable to the geometry of a thermoelastic shell. By accounting for the coupling of mechanical and thermal fields, the resulting equations accommodate the thickness, extensional, flexural and torsional as well as coupled motions of thermoelastic shell at both low- and high frequencies. Also, a theorem of uniqueness is devised in solutions of the two-dimensional, initial-mixed boundary value problems defined by the system of shell equations, and the boundary and initial conditions sufficient for the uniqueness of solutions were enumerated as well.

The unified variational principle (41) which allows one to make simultaneous approximation upon all the field variables provides a standard basis in directly generating various numerical solutions for the thermo-mechanical analysis of an elastic medium. This differential type of variational principles is in general agreement with the variational principles reviewed and also reported by Altay and Dökmeci (1996). The hierarchic system of two-dimensional equations was shown to recover various types of shell and plate equations provided the thermal effect with second sound or the temperature-dependency of material and/or the effect of curvature were abrogated. By omitting the coupling effect in the constitutive relations, one arrives at the hierarchic system of uncoupled equations, in which the mechanical and thermal responses of shell can be separately treated. The resulting equations of thermoelastic shell can be further simplified by introducing a variety of properties involving the geometrical configuration, kinematics and material property of thermoelastic shell of uniform thickness.

Noteworthy is the fact that some assumptions involving stress or strain distributions across the shell thickness together with the compatibility conditions may be taken as an alternative basis in lieu of the fields of displacement and temperature increment in deriving the hierarchic system of shell equations. Besides, Mindlin's method of reduction can be readily replaced by the direct integration method of reduction (Naghdi, 1972) or the asymptotic geometric optics method (Steele, 1965) so as to deduce the equations of thermoelastic shell from the three-dimensional equations of thermoelasticity. Although, the hierarchic system of equations which is exclusively formulated for the high-frequency vibrations of a thermoelastic shell, it is rather straightforward to extend it to that of a shell with time-and/or temperature-dependent types of materials (polar, porous, nonlocal and alike), with large strains (Başar and Ding, 1997; Altay and Dökmeci, 2000), and with the inclusion of moisture (Doxsee, 1989) and piezoelectric effect (Dökmeci, 1974).

Evidently, the hierarchic system of two-dimensional equations of a thermoelastic shell is approximate but more tractable for numerical computation than the system of three-dimensional fundamental equations of thermoelasticity expressed for a shell. Both the systems of thermoelastic shell equations inherently contain some errors of experimental nature due to the constitutive relations. The constitutive type of experimental errors cannot be reduced by simply increasing the accuracy of computation in solutions of the system of two-or three-dimensional shell equations. In addition, both the systems of shell equations have some inevitable errors in engineering applications, arising from the prescribed boundary and initial conditions and the rate and type of loading as well as the method of numerical computation. Besides, the hierarchic system of shell equations includes some kinematic errors depending on the order of truncation N in Eq. (52) and also Eq. (60), and in particular, on the shell parameter (42). The relative merits of using the two-dimensional system of thermoelastic shell equations in lieu of the three-dimensional ones depend obviously on specific applications. Thus, it may be concluded that it is a necessity to carry out research involving error estimates in solutions (John, 1965) of both the systems of two-and three-dimensional equations of thermoelastic shell.

In closing, by the use of the hierarchic system of coupled dynamic thermoelastic shell equations, certain applications, especially including the effect of hyperbolic heat conduction, which is beyond the scope of this paper are the topics of follow-up papers. Besides, some extensions of the hierarchic system with special emphasis on numerical algorithms based on the method of moments (Dökmeci, 1992) for a class of applications will be reported elsewhere. Lastly, the demonstrations of the existence of solutions, especially with reference to the specifications of general boundary and initial conditions (Bernadou et al., 1994) and the convergence properties of the hierarchical system of thermoelastic shell equations (Oliveira, 1974; Antman, 1997) remain to be investigated as well.

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